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Оригинални научни рад

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The Equals Sign – the Problem of Early Algebra Learning and How to Solve It

Summary: *The equals sign is one of the most important concepts and symbols in mathematics, the understanding of which is crucial for learning and understanding all mathematical content, especially algebra. The paper will draw attention to the problems related to the formation and understanding of this concept, and present a methodological approach to learning based on the context modeling of real-life situations (modeling length, balance, etc.), with the aim of overcoming this misconception. In the empirical section of the paper, we examined the effects of the methodological framework of learning algebra on the understanding of the equals sign through an experiment with parallel groups, and on a sample of the fourth-grade students of primary school (N = 257). The obtained results show that the methodological approach based on the context modeling of real-life situations improves student understanding of the equals sign, as a symbol of mathematical equivalence and their ability to solve problems containing this sign.*

Keywords: *contextual approach, equals sign, early algebra, Realistic Mathematics Education, student achievement*

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Introduction

One of the most important concepts and symbols in mathematics is the *equals sign*. This concept is formed simultaneously with the formation of the concept of natural numbers, and learning of the first rules and laws of arithmetic. The correct understanding of this concept is crucial for learning and understanding all mathematical content, especially algebra. However, a large body of research indicates various problems that accompany the correct understanding of the equals sign (Booth, 1988; Carpenter & Levi, 2000; Filloy & Rojano, 1989; Freiman & Lee, 2004; Kieran, 1981; Knuth et al., 2006; Marjanović, 2002; McNeil & Alibali, 2005; Sfard & Linchevski, 1994), which are later reflected on learning algebra. The importance of the equals sign in mathematics is best metaphorically summed up by Jones, who says: "An arithmetic expression is like a film set on which the numbers are actors, the operators are the script and the equals sign the director who shouts *Action*!" (2006: 6). However, the concept of "equality is a central – but sorely neglected – concept in mathematics education" (Parslow-Williams & Cockburn, 2008: 35).

Theoretical Foundations of the Research

The difficulties related to the misunderstanding of the equals sign in algebra stem from the precarious foundations on which the said concept is formed in arithmetic learning. Specifically, students understand this sign as operational in arithmetic, i.e. as a sign that signifies the command "calculate" or "ascertain", instead as a relational sign that signifies "equivalence" (Baroody & Ginsburg, 1983; Ilić, Zeljić, 2017; Kieran, 1981; McNeil & Alibali, 2005). This is confirmed by Parslow-Williams and Cockburn who notice that "children see '=' as an instruction to complete an operation" (Parslow-Williams & Cockburn, 2008: 36). Research by Rittle-Johnson et al. indicates that evidence for this has primarily come from three different classes of equivalence tasks: (a) *equation-solving items*, such as $8 + 4 = \_\_\_\_\_ + 5$; (b) *equation-structure items*, such as deciding if $3 + 5 = 5 + 3$ is true or false; and (c) *equal-sign definition items* (Rittle-Johnson, et al., 2011).

Learning arithmetic, students acquire the habit to observe the left side of the equals sign as the side that always represents the request for performing an operation, while the right side solely represents the result. In consequence, students at this age often believe that equations, such as $8 = 3 + 5$, are incorrect (Kieran, 1981; Filloy & Rojano, 1989; Carpenter & Levi, 2000). Similarly, McNeil, et al. (2010) argue that even high school students interpret the equals sign differently in "non-standard" equations (e.g. $3 + 4 = 5 + 2$ and $7 = 7$) with regard to "common" equations to which they are accustomed (e.g. $3 + 4 = 7$).

Students' first reading of the equals sign is often in the context of the instructions for performing the arithmetic equation, for example $3 + 5 = ?$. This reading of the 'process' of the equals sign often reads as follows: *3 plus 5 gives 8*. When this process is later understood as a number sentence, the equals sign often acquires additional meaning, such as *is*, or *is the same as*, or even *equals*. Reading the equation $3 + 5 = 8$ then becomes a number sentence which reads *3 plus 5 is 8* or *8 is the same as 3 plus 5*, which is an important conceptual change.

The first step in developing the equals sign is to have students understand the concept of a *mathematical expression* as an *object*, i.e. an independent whole. This means that students should not only be able to see the expression as a process of performing an operation, but to understand that it can exist on its own (Freudenthal, 1962; Kieran, 1981, 1992; Sfard, 1991; Sfard & Linchevski, 1994; Zeljić, 2014). Thus, from the mentioned research, Dabić Boričić and Zeljić (Dabić Boričić and Zeljić, 2021) notice that if expressions are understood "as processes (calculating the value of expressions), and not as objects with a meaning of their own, students will understand algebraic expressions as evaluation procedures, instead of mental entities that can be manipulated" (Dabić Boričić and Zeljić, 2021: 31). For example, when encountering the expression *8 + 5* for the first time, a student's first instinct will be to perform the operation, i.e. to determine the value of the expression and write down the following *8 + 5 = 13*. However, to learn algebra, it is crucial that students understand and observe the expression *8 + 5* as a whole which can exist on its own and can be manipulated, same as numbers. Only when the students are able to understand a mathematical expression as an independent object can they reach structural understanding and deeper understanding of the expression and thus master the concept of equality. In this regard, the equals sign has a key role as a symbol that enables such manipulations. However, research shows that many students lack the flexibility needed to approach the mathematical structure of the expression and equations (Van Stiphout et al., 2013). If the meaning of this concept is not formed properly, one's understanding of the equals sign will always be reduced to the expectation that the result of a mathematical expression is always located on the right side of the equals sign (Booth, 1988). Therefore, an important task of learning arithmetic is to draw more attention to the formation of the equals sign concept. Research shows that using appropriate content, created specifically for this purpose, can help in the correct formation of the equals sign as a symbol of equivalence (Dabić Boričić i Zeljić, 2021; McNeil et al., 2015; Parslow-Williams & Cockburn, 2008). In modified circumstances where students solve simplex expressions which involve addition and subtraction, for example ____ *= 4 + 3*, the equals sign should be replaced by words denoting equivalence: "is equal to", "is the same as", "something is equivalent to something else", etc.

The solution to this problem can be found in the use of visual representations, as noted by Dabić Boričić and Zeljić (Dabić Boričić and Zeljić, 2021), because they encourage structural conception, making abstract ideas more tangible (Fagnant & Vlassis, 2013). As a form of visual representation, schemes are representations comparable to the drawings students make when solving textual (word) problems, and have a positive effect on solving mathematical problems (Fagnant & Vlassis, 2013). We provide several methodological solutions that were applied in our research which refer to the use of visual representations.

Let's take a simple example of *by adding 3 to 2, I will get 5*, using counting sticks that illustrate this process as a demonstrated equivalence (Figure 1). By naming the equivalence as *2 plus 3 is the same as 5 (length/size/sum),* we begin to establish a number sentence and the concept of equivalence inherent in the equals sign.

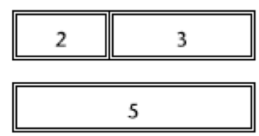


Figure 1. Equality of lengths 2 + 3 and 5
(Ryan, Williams, 2007: 27).

In this way, students are introduced to the concept of mathematical equivalence – the principle that the two sides of an equation represent the same value. Understanding mathematical equivalence is an indicator of a student's flexibility in presenting and using basic arithmetic ideas, and an important requirement for understanding algebra (DeCaro & Rittle-Johnson, 2012; Knuth et al, 2006; Rittle-Johnson et al., 2011). Based on their early experience in solving mathematical problems where the equals sign is located at the end of the equation (e.g. *3 + 5 = x*), students often mistakenly conclude that the equals sign is a symbol that indicates a "sum", or "provide an answer". Therefore, when solving problems which involve operations on both sides of the equation (e.g. *3 + 5 = x + 2*, the problem of mathematical equivalence), students often add up numbers on the left side of the equation (E.g. answer is "8"), or add up all numbers (E.g. the answer is "10") in keeping with their misconception about the meaning of the equals sign. DeCaro and Rittle-

Johnson imply that "second- to fourth-grade students have the computational skills to solve arithmetic problems but generally have little prior experience in solving equivalence problems" (DeCaro & Rittle-Johnson, 2012: 555).

Parslow-Williams and Cockburn propose an "undo" strategy for the formation of the equals sign (Figure 2). The strategy is based on the materialization of mathematical notation and manipulative activities of borrowing and regrouping in subtraction.

The large-scale presence and breadth of meaning of the equals sign at all levels of mathematics education underscores its importance. The concept of the equals sign and its symbolic meaning is introduced in the junior grades of primary school, while in senior grades this concept is given very little attention. Given the cognitive level of the junior primary school students and their reliance on the concrete and the obvious in the learning process, the formation of this concept should be based on specific manipulative activities which are carried out by students through the use of multiple visual representations. This primarily involves the use of visual and symbolic representations, and less the use of verbal expressions and diagrams (Alexandrou-Leonidou & Philippou, 2011). In this way, organized learning has the characteristics of cognitive growth and becomes more than a simple process of arranging mathematical knowledge in a student's mind. The learning process is based on one's independent construction of mathematical knowledge through the process of mathematization, and based on problems rooted in real-life contexts. That being said, the idea is that the development of this concept at junior primary school age should be based on learning situations in which a student models a real-life situation, solves problems with real context, whereby a real-life situation expresses equivalence. This attitude is confirmed in the research by Dabić Boričić and Zeljić who conducted an experimental research with the fourth-grade students of primary school, showing that "the modeling process can positively impact the construction of meaning and understanding of equivalent forms of expressions" (2021: 32). These situations must be experientially close to the student. Such activities can be observed as a way of rediscovering mathematical ideas, but in one's own way, with the aim of acquiring individual knowledge for which students themselves are responsible (Sarama & Clements, 2009). Students are challenged to independently build strategies in the process of problem-solving by thinking about the problem and discussing it with others. In this way, students develop their own informal strategies and the teacher's task is to enable a gradual formalization of these strategies so that they could be used in other similar situations as well (Milinković, 2021).

Here are some examples of the context modeling of real-life situations that can lead to the formation of the concept of equality. For example, students can be familiarized with the concept of equality through modeling, whereby ribbons of equal

6 – 2 = 9 – ?

////XX ////XXXXX

? – 3 = 7 – 5

//XXX //XXXXX

5 = ? – 1

////X

Figure 2. Pupils' recording strategies for solving equality problems
(Parslow-Williams and Cockburn, 2008: 28).

length are used (Figure 3). This example illustrates a real-life context, because students have the opportunity to observe the length of ribbons and make conclusions about the relations between them. Students will first verbally express the relations: *The blue ribbon has the same length as the yellow and purple ribbons combined*, or *The blue ribbon has the same length as the purple and green ribbons combined.*

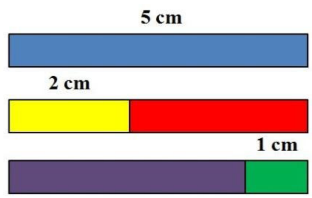


Figure 3. Illustration of the model with ribbons of the same length.

In the next step, students' thinking moves from the real world into the realm of formal mathematics, through the process of horizontal mathematization, thus enabling the notation of mathematical equality. Students are now able to express the equality of the ribbon lengths in mathematical language. Based on the given situation, they will write down the following equations, whereby unknown lengths will be written on the placeholders:

5 cm = 2 cm + ____ cm

5 cm = ___ cm + 1 cm

2 cm + 3 cm = ____ + 1 cm etc.

Based on the equations above, the student notices the value of the unknown part. The ratio between the lengths of different ribbons can be expressed as an equation with unknowns:

5 cm = 2 cm + ___ = ____ + 1 cm

In this way, a real-life context for understanding the equals sign as a symbol of equivalence was created – what is located on the left side of the equation is equivalent to what is located on the right.

Horizontal mathematization enables the transition from the real world into the realm of abstract mathematical notation which is used to express the ratio between lengths. Furthermore, the student has the opportunity to compare expressions on both sides of the equals sign and determine the lengths based on that. Given that the equals sign expresses equivalence, it means that the value of each of the three expressions separated by the equals sign must be the same. Thus, in the process of vertical mathematization, the student will conclude that the solution of the equation must be the following:

5 cm = 2 cm + 3 cm= 4 cm + 1 cm

Finally, the student returns to the real world and explains the results he/she obtained, stating that the length of the red ribbon is 3 *cm,* whereas the length of the purple ribbon is 4 *cm*. In practice, omission of the unit of measurement, in this case length, is achieved gradually and reintroduction or reconnection of the model with the numerical expression may be needed to build a chain of meaning over a long period of development of this concept.

Problem modeling using the "scales" can contribute to the understanding of the concept of the equals sign. In this way, we gradually build up the meaning of the equals sign as a symbol of equivalence, because these problems revolve around the notion of balance that exists between two sides of the scales. The model used in solving this type of problems is based on the notion of equal quantities on both sides of the equals sign, and is known as the balance model or the scales model. The following example is based on the scales model (*Figure* 4).

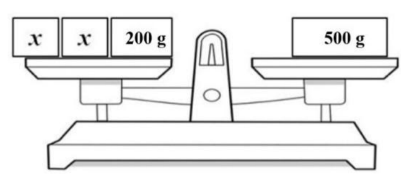


Figure 4. Situation modeling using the scales – unknown on the left side.

Students are able to observe the mass of the weights placed on the pans of the scales. In addition, since the scales are in balance, it means that the mass of the left pan equals the mass of the right pan of the scales. In the process of mathematization, the real-life context of the mass of weights can be translated into the following equation: $x + x + 200\ g = 500\ g$. In this form, the mathematical equation expresses the balance of the scales in reality, so that the concept of *equality* can be understood as a concept that expresses the balance of elements located on the left and the right side of the sign – *equal, equivalent, in balance*. Thus, the process of solving the equation, which can be written down, can be reduced to manipulating the scales, whereby the aim is to keep them in balance.

The situation can also be modeled so that the unknown is located on the right side of the equals sign, which is not common, nor something that students are accustomed to (*Figure 5*).

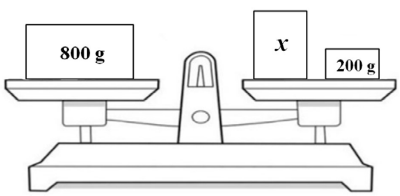


Figure 5. Situation modeling using the scales – unknown on the right side.

The process is also expressed through the comparison of equal masses on both sides of the scales. The real-life context would be translated into the following equation $800\ g = x + 200\ g$ in the process of mathematization. Examples like this can be a good basis for correct understanding of the equals sign as a symbol of equivalence.

Here are just some examples how real-life situations and their modeling can be used for the purpose of developing the concept of equality. These situations can be modeled by using specific sets formed of didactic material that students can manipulate. The ability to correctly understand the equals sign implies a student's ability to understand it as a symbol that expresses symmetry between the left and the right side of the equation. The goal is to redirect students' thinking from the primary understanding of the sign as a command "calculate" toward understanding the sign as an expression of "equivalence". In this way, students are able to understand symbolic expressions as processes that exist as independent objects.

Methodological Framework

Based on the ideas above, the aim of this paper is to examine whether the methodological approach based on the real-world situation modeling contributes to a better understanding of the equals sign and thus to an improved learning of early childhood algebra in which this concept features prominently. Based on the research aims, two research tasks were operationalized:

1. Examining whether the real-world situation modeling affects the understanding of the equals sign as a sign that expresses the equivalence of mathematical expressions.
2. Examining whether students use different forms of visual-schematic representations to express the equivalence of mathematical expressions in solving algebraic problems, i.e. in the process of problem presentation.

The research sample was selected from the population of students who attended the fourth grade of primary school during 2019/2020 in Serbia. Ten classes of fourth-graders (N = 257) from three primary schools in Užice participated in the experimental research. The experimental group comprised students (N = 130) from five classes, all from the same school. Students from the control group (N = 127) came from two primary schools (three classes from one school and two from the other). The groups were homogenized for the research purposes based on the overall achievement and grade point average in mathematics at the end of the 3rd grade, whereas their homogeneity was controlled through the analysis of covariance.

The research was carried out by applying an *experimental method*, specifically, an experimental model with parallel groups. The program followed by the experimental group comprised specially prepared lessons where algebra was taught using real-life context modeling. The class load was 25 lessons/classes, whereby students from the control group learned in a traditional way, using the approved mathematics textbook for the 4th grade of primary school. In the control group, learning was based on introducing the content of early childhood algebra in a mathematical context, without the real-world situation modeling. In the traditional teaching approach, the teacher first introduced the terminology and continued by presenting two or three problems, inviting all students to solve them; the students then tried to solve the given problems (tasks) from the course book (textbook) using the learned terminology; finally, the teacher highlighted several problems from the course book and asked the students to do the tasks from the textbook. In the contextual approach, the starting point in the learning process was a problem expressed in the form of visual representation, words, or another type of designation. Relationships that represent mathematical problems rooted in reality or the students' environment were expressed by shaping a problem situation. The problem situation was then modeled in different ways through the process of horizontal mathematization and expressed symbolically.

Early algebra content in the experimental program included: equations with addition, subtraction, multiplication and division; relation between the changes of the result and the changes of the computational components. Instruction in both groups was carried out according to the regular school curriculum. The research utilized the *test method*. Two equivalent tests were created for research purposes: initial and final test. The test contained the tasks that aimed to examine 1) understanding of the equals sign of mathematical expressions expressed in a purely mathematical context, and in the context of a problem situation; 2) a real-world problem in the field of algebra that aimed to examine whether students use different forms of visual-schematic representations that express the equivalence of mathematical expressions in solving algebraic problems, i.e. in the process of problem presentation. The tasks in these tests were real-world context problems and included problems in which the correct use of the equals sign was the key for solving them, such as balancing tasks and line segment modeling tasks. In addition to these tasks, the test also included mathematical context tasks aimed to examine the students' understanding of the equals sign as the symbol of equivalence of mathematical expressions. We will include several examples of the test tasks expressed in the real-world and mathematical context below.

Examples of the real-world context tasks:

1. Determine the mass of the apple by looking at the picture.

The mass of the apple is: ____________.

2. Three roses cost the same as two roses and two tulips. How much does a rose cost, if a tulip costs 300 dinars?

Examples of mathematical context tasks

1. Fill in the missing numbers to make the equation correct.

   30 + 40 = __ − 50 = __ + 60 = __

2. Fill in the missing numbers to make the equation correct.

   350 + __ = __ · 20 = __ : 2 = 700

Before the experiment, initial testing was performed in both groups to determine the initial level of understanding of the equals sign. After the initial testing, the experimental program was introduced in the experimental group, and implemented in regular classes for the 4th grade. Upon completion of the experimental program, final testing was performed. Tests comprised math problems, where each was worth 5 points. The scores were determined depending on whether each problem was solved in its entirety, or only partially.

The values of Cronbach's alpha for the initial and final tests indicate results greater than 0,7, which indicates an acceptable and desirable reliability of the constructed tests. The values of skewness and kurtosis for the initial and final tests in the groups range between -2 and 2 (Table 1), meaning that the results show a univariate normal distribution (George & Mallery, 2010).

Given that the sample size in the research is greater than 200, this distribution of the results does not have a large effect on the results of the analysis (Tabachnik & Fidell, 2013).

The obtained results were analyzed *quantitatively* and *qualitatively*. When it comes to statistical measures, the analysis of variance and analysis of covariance (ANCOVA) were used to determine the existence of the potentially statistically significant differences in achievement between the experimental and the control groups, and thus monitor the effects of the experimental program. The analysis of covariance was also used to statistically control the homogeneity of the experimental and the control group by removing variance in the dependent variable.

Results and Discussion

The initial measurement of the student ability to correctly understand the equals sign shows that both student groups achieve poor results, namely: experimental (M = 4.07; SD = 2.71) and control group (M = 3.87; SD = 2.47) (Table 2.). Given that the maximum number of points was 10, the obtained results show that students failed to even score a half of the maximum number of points in the test that measured the ability to understand the equals sign in the initial measurement. These results indicate that the concept of the equals sign is not fully developed in either group (experimental and control).

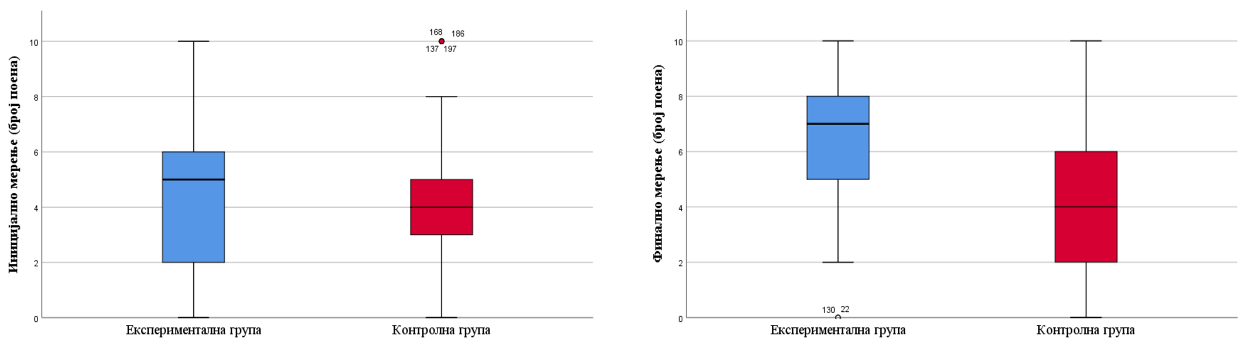
Students from the experimental group achieved better results in the final measurement (M = 6.52; SD = 2.50) with regard to the initial measurement, in contrast to students from the control group who scored approximately the same results as in the initial measurement (M = 4.02; SD = 2.41), which are still below average (Graph 1). It is obvious that the students who adopted algebra content according to the principles of the contextual approach made greater progress than students who worked in line with the traditional approach to teaching mathematics.

Table 1. Testing the normality of the initial and final tests.

Group	Test	Skewness	Sde	Kurtosis	Sde
Experimental group	Initial test	.177	.212	- .825	.422
	Final test	- .278	.212	- .635	.422
Control group	Initial test	- .166	.215	- .655	.425
	Final test	- .141	.215	- .938	.427

Table 2. Descriptive indicators of student achievement in the experimental and control groups in the initial and final measurement.

		N	M	SD
Initial test	Experimental group	130	4.07	2.71
	Control group	127	3.87	2.47
Final test	Experimental group	130	6.52	2.50
	Control group	127	4.02	2.41



Graph 1. Results of the control and experimental group in the initial and final measurement.

Differences in achievement between students from the experimental and students from the control group in the ability to correctly understand the equals sign were tested using analysis of variance (Table 3). The values of Levene's test in the initial measurement (F(1,255) = .889; p = .347) and the final measurement (F(1, 255) = .169; p = .681) show that the assumptions about the homogeneity of vari-

Table 3. Analysis of variance in the initial and final measurement.

	Levene Statistic	df1	df2	Sig.
Initial test	.889	1	255	.347
Final test	.169	1	255	.681

		Sum of Squares	df	Mean Square	F	Sig.
Initial test	Between Groups	2.448	1	2.448	.364	.547
	Within Groups	1716.361	255	6.731		
	Total	1718.809	256			
Final test	Between Groups	401.333	1	401.333	66.482	.000
	Within Groups	1539.360	255	6.037		
	Total	1940.693	256			

ance are not violated, therefore, the results obtained by using the analysis of variance can be considered reliable.

The value of analysis of variance obtained in the initial measurement (F (1, 255) = .364; p = .547) shows that there is no statistically significant difference between the students from the experimental and the control groups with regard to the correct understanding of the equals sign. In contrast, the value of the analysis of variance in the final measurement (F (1,255) = 66.482; p < .000) shows that there is a statistically significant difference between groups with regard to the measured ability to correctly understand the equals sign. The obtained results show that students who adopted algebra content through modeling of real-life situations made better progress in the ability to correctly understand the equals sign with regard to students from the control group.

To eliminate the suspicion of group non-homogeneity, we conducted the analysis of covariance (ANCOVA). Levene's test of equality of variance equals F = 2.853 for p = .092 meaning that the assumption of equality of variance is not violated (Table 4).

Table 4. Leven test of homogeneity of variance.

Dependet Variable: Final measurement			
F	df1	df2	Sig.
2.853	1	255	.092

Tests the null hypothesis that the error variance of the dependent variable is equal across groups.
a. Design: Intercept + Equality1 + Group

By analyzing covariance and eliminating the effects of covariates (initial measurement of the ability to correctly understand the equals sign), we can conclude that there is a statistically significant difference in achievement between students from the experimental and the control group (F (1, 254) = 143.84; p < .000) (Table 5). In this case, partial eta squared is .362, which is a large effect according to Cohen (Cohen, 1988). This means that as much as 36.2% of variance in the final measurement can be explained by the independent variable, or in this case, the implemented model of learning algebra in mathematics education. If the effect of covariates (results in the initial measurement of the ability to understand the equals sign) in the final measurement of the same ability is considered, and if the effect of the independent variable is eliminated (group), we will nonetheless get statistically significant differences in the ability to correctly understand the equals sign (F (1,254) = 367.14; p < .000). The value of partial eta squared is 0.59, i.e. a large effect (Table 5).

The results of statistical analysis show that context modeling of real-life situations in algebra learning contributes to the correct understanding of the equals sign. This is especially important if we consider the role the equals sign plays in algebra. Inability to understand the equals sign is primarily an obstacle to understanding arithmetic, but also to mastering and understanding algebra content – *equations, functional dependency,* etc.

We will list a few examples of errors we identified in students' answers in the test for illustrative purposes which indicate their lack of understanding of the equals sign (Figure 6). The task students had to solve was: *Determine the mass of the apple based on the picture.*

Figure 6. Solving a balance task – inability to understand the equals sign.

Table 5. Analysis of covariance between the initial and final measurement.

Dependet Variable: Final measurement						
Source	Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared
Corrected Model	1311.209[a]	2	655.604	264.540	.000	.676
Intercept	433.594	1	433.594	174.957	.000	.408
Initial measurement	909.876	1	909.876	367.140	.000	.591
Group	356.477	1	356.477	143.840	.000	.362
Error	629.484	254	2.478			
Total	9127.000	257				
Corrected Total	1940.693	256				

a. R Squared = .676 (Adjusted R Squared = .673)

The student successfully found the solution, but the equation he used in the process of solving the task was incorrect. In this case, the equals sign connects several expressions the values of which are not the same. In solving this task, one can recognize student's correct thinking in the process (determining the mass on the right plate of the scales to indicate the total mass on the right plate). The error occurs through an inadequate use of the equals sign, because the student understands and uses it as "calculate value" instead of "it is equivalent to". This error, and other similar ones identified in our research, confirms the results of other research (Mason, 1996; Kieran, 2006; Obradović and Zeljić, 2015) which indicated a tendency among a certain number of students to use the equals sign incorrectly, i.e. to focus on the result instead on the solution process and representation of the mathematical concept.

Many students demonstrated a lack of understanding of the equals sign as a symbol that expresses equivalence. Instead, they relate this concept to arithmetic and equate it with commands "determine value" or "calculate". Students had the following task: *Fill in the missing numbers: 30 + 40 = ____ – 50 = ____ + 20 = _____.* An example of the most common error in solving this problem is shown in Figure 7.

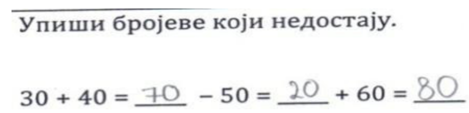
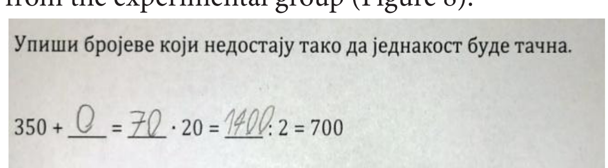


Figure 7. Solving a task in a mathematical context.

Kieran also argues that in these situations, "school children, despite efforts to teach them otherwise, view the equals sign as a symbol which separates a problem and its answer" (Kieran, 1981: 324). The example shows that students do not observe the whole equation, but are solely focused on individual expressions and their value instead. The same example also shows that students did not yet form the concept of *expression*. Moreover, in solving the task, students perform operations neglecting equivalence relations between the expressions. In the final measurement of the effects of the experimental program, students were given a similar task: *Fill in the missing numbers to solve the equation: 350 + ____ = ___ · 20 = ___ : 2 = 700.* Students from the control group yet again demonstrated a lack of understanding of the equals sign, unlike students from the experimental group (Figure 8).



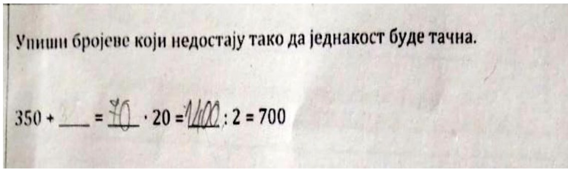


Figure 8. Final test done by students of the control group.

The examples above show that students see the equals sign solely as the "calculate" command, instead as a symbol of equivalence. The incorrect way in which students solved the task also tells us how they approached solving the task, i.e. by starting from the known values in the equation. Only the values of the expressions from the right are correct, while the values of all other expressions arise from the lack of understanding of the equals sign, which led to incorrect values in other expressions as well. Students from the experimental group, however, successfully solved this problem.

When it comes to solving real-life problems from the field of algebra, students from the experimental group made progress in solving these tasks. We will list a few examples to illustrate their progress in understanding the equals sign, and the ability to solve problems containing this sign. The special benefit of the implemented contextual methodological approach and proof of its effects is reflected in the fact that students also adopted an approach to solving problems based on presenting problems in different ways using various visual-schematic representations that aim to express the equivalence of mathematical expressions.

The students were given the following problem in the test: *Three roses cost the same as two roses and two tulips. What is the price of a rose, if one tulip costs 300 dinars*? Students used visual representations in solving this task to make it more relatable and highlight the relations between data in the given task (Figure 9).

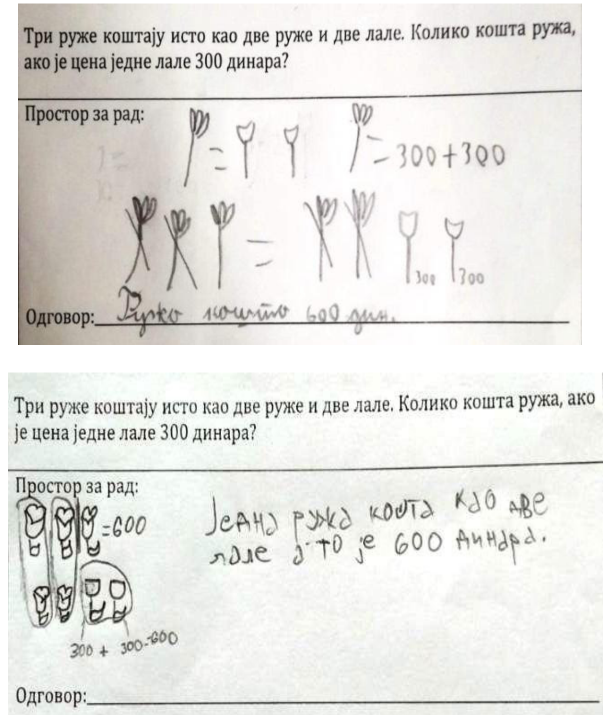


Figure 9. Solving a task – students from the experimental group.

In this case, the pictogram served as a model in problem solving, and it was obtained through the mathematization of a real-world context given in the task. Students used the pictogram to express contextual situations, whereas the equals sign is used to determine relations that exist in the task. In this case, we can conclude that students fully understand the equals sign as a symbol of equivalence, because they eliminated identical values on the left and the right side of the equals sign in the process of solving the task. The process of eliminating identical values on both sides of the equation is the basis which can later be used as a procedure for solving equations that avoids the second unknown, and some authors used it as a strategy for forming the concept of the equals sign (Parslow-Williams & Cockburn, 2008).

A certain number of students solved the same problem using the line segment model as an ideogram, to express relations in the task (Figure 10).

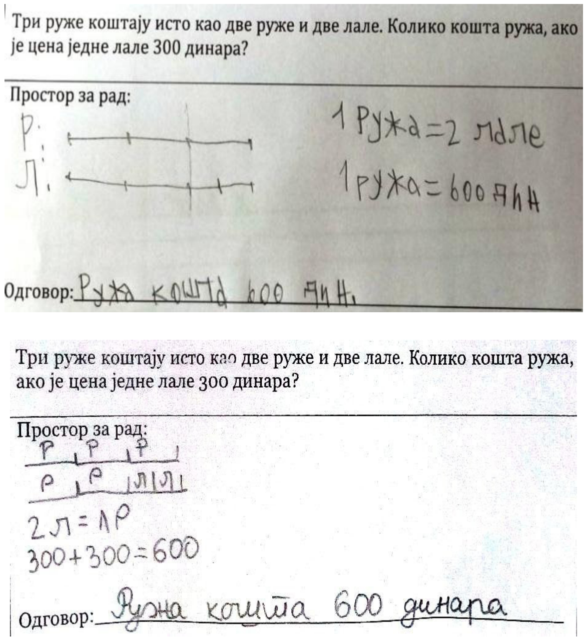


Figure 10. Solving a task by line segment modeling.

Unlike the solution where the pictogram directly expressed the relations between prices of different flowers, in this case, students used a more abstract model, ideogram (line segment model). When we observe the way in which the students solved this problem, we recognize the tendency toward problem visualization, whereby the length of the line segment represented the price of roses and tulips as data given in the task. The line segment model was used to understand the relations between known and unknown values in the task more fully, but also to facilitate its solution. The above examples of solutions clearly show the students' need to use models to express relations, but also to avoid their symbolic notation. This is a direct consequence of the approach used in the experimental group, i.e. mathematical modeling of real-life situations and presenting them through visual representations to ensure a better understanding and shifting the context from mathematical models to real-world contexts. The reason for this can be found in an inherent need for clarity in children of this age. Obradović and Zeljić, who believe that using models to represent information should lead to correct reasoning and improved relation-awareness, obtained similar results (2015). The authors especially underline that "without the use of different representations and models, students cannot always understand the structure of a task, or the procedure of its solution" (Obradović & Zeljić, 2015: 77). The study by Otten et al. (2020) showed that students who use balance problems in learning algebra later use representations, models and advanced algebraic strategies more often to learn and solve mathematical problems. The results of the said study show that different model representations play a very significant role in the process of learning algebra content.

Conclusion

The equals sign – one of the basic concepts in arithmetic and algebra, an important mathematical symbol, and a "deep understanding of the notion of equality is a necessary prerequisite for primary school students in order to proceed to upper school mathematics" (Alexandrou-Leonidou & Philippou, 2011: 410). The root of incorrect and insufficient understanding of the equals sign lies in its formation in arithmetic learning, whereby the same problem is later transferred to the field of algebra. The fact is that the correct construction of this concept should begin in arithmetic (Marjanović & Kalajdžić, 1992), but in that process, the focus should be on real-life situations which will familiarize students with the equals sign and help them understand it as a symbol of equivalence.

The central issue in understanding the equals sign is to avoid observing this concept as operational, but to develop its relational meaning instead. Understanding the sign "=" as operational causes difficulties with its relational meaning among students. The results obtained in our experimental research show that by shaping mathematical content so as to be based and modeled in real-life learning situations, we can positively influence students' ability to correctly understand the equals sign. Van Reeuwijk (2001) came to the same conclusion, pointing out that a real-life context helps students to understand the meaning of formally expressed mathematics and the equals sign as a symbol of equivalence, and thus gain a more comprehensive understanding of the concept of equation. This means that a methodological approach to learning, based on the principles of Freudenthal's phenomenology, which involves modeling learning situations that express clear and real situations from everyday life through the processes of horizontal and vertical mathematization, is a major driver behind the correct understanding of the equals sign. In that process, different representations based on extreme clarity and obviousness should be used so as to help students understand equality as equivalence. A special role in this process belongs to balance tasks which express mathematical equivalence in an obvious way, enabling a real-life context to serve as a foundation for building a deeper meaning of the equals sign. These tasks can also be used as a model for solving equations in the process of formal problem solving, by performing the same operations on both sides of the equals sign (Vlassis, 2002). Dabić Boričić and Zeljić (Dabić Boričić & Zeljić, 2021) obtained similar results in their research with students aged 10 and 11. The results of this research show that students are capable of developing a correct understanding of the equivalence of mathematical expressions, regardless of whether these are arithmetic or algebraic expressions. The authors believe that the key to success lies in the modeling process and different representations, meaning that the modeling process should not be used exclusively as a framework for solving textual (word) problems in teaching, but also for developing the meaning of algebraic concepts.

The results of our research show that students use different forms of visual-schematic representations in solving real-world context algebraic problems to present the equivalence of mathematical expressions. Working on such tasks not only enables students to develop a deeper meaning of the concept of equality, but also contributes to a more successful solution of algebra problems. Correct understanding of the equals sign in arithmetic is a crucial ability which helps one lay the foundations for later learning of algebraic content, and the key period for the development of this ability in mathematics education is junior primary school. For these reasons, this mathematical concept should be approached carefully, and gradually developed through the curriculum (Alexandrou-Leonidou & Philippou, 2011; Alibali et al., 2007; NCTM, 2000), because, otherwise, we have to be aware that a limited understanding of the equals sign (operational meaning only) can hinder the development of algebraic concepts. Practitioners should be particularly mindful of this fact when designing mathematical activities, starting from the initial idea of the concept of equivalence (comparing numbers), then moving to the role of the equals sign in the formation of the concept of operations, and finally expanding the meaning of the sign to the equivalence between the left and the right side of the equation, instead of simply understanding this concept as the command *Calculate!*. All these activities should be based on distinctly practical activities which allow students to create situations to express the equivalence between the two sides of the equation. In this way, we will divert students from over-relying on patterns and equalities in purely mathematical form, and provide them with an opportunity to construct a conceptual understanding of the equals sign from situations that express real-life contexts.

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ЗНАК ЈЕДНАКОСТИ – ПРОБЛЕМ УЧЕЊА РАНЕ АЛГЕБРЕ И КАКО ГА РЕШИТИ

Један од најзначајнијих појмова и симбола у математици јесте знак једнакости. Правилно разумевање овог знака је један од најзначајнијих предуслова за разумевање и учење других математичких и алгебарских појмова. У раду се посебно истичу потешкоће које прате неправилно разумевање знака једнакости, чија се основа проналази у неправилном формирању овог појма у настави аритметике, где се ученици навикавају на знак једнакости као знак који означава „израчунај" или „одреди". Најзначајније за разумевање знака једнакости јесте управо то да се овај појам не схвата само операционо, већ да се развије релационо значење овог симбола. Разумевање знака „=" само операционо доводи до тога да ученици имају потешкоће са разумевањем његовог релационог значења. Имајући то у виду, аутори у раду истичу и развијају идеју да на млађем школском узрасту учење и формирање знака једнакости треба да буде засновано на ситуацијама учења у којима ученик моделује реалне ситуације и решава проблеме реалног контекста. На овај начин проблеми реалног контекста одражавају реалне ситуације из свакодневног живота које упућују на суштину појма знака једнакости - као знака који изражава еквивалентност. У складу са тим у раду је представљен методички приступ учења заснован на моделовању ситуација реалног контекста како би се проблем неразумевања појма знака једнакости превазишао. У моделовању ситуација реалног контекста посебно је издвојено моделовање дужина и равнотеже, јер се на овим моделима најлакше уочава и схвата еквивалентност као основа за изградњу *појма знака једнакости у математици. Приказани методички приступ експериментално је тестиран у оквиру истраживања које је имало за циљ да испита да ли овакав методички приступ доприноси бољем разумевању знака једнакости. Кроз експеримент са паралелним групама на узорку ученика четвртог разреда (N=257) реализовано је истраживање, у оквиру кога су испитивани ефекти методичког приступа. Програм по коме је радила експериментална група чинили су посебно припремљени часови (25), у којима се користило моделовање ситуација у реалном контексту, док су ученици који су чинили контролну групу радили на уобичајени традиционални начин, користећи одобрени уџбеник из математике за четврти разред основне школе. Добијени резултати показали су да се релационо значење знака једнакости може развијати употребом проблема који изражавају ситуације реалистичног контекста. Резултати добијени у овом истраживању показали су да се пажљиво обликованим садржајима математике, тако да они буду засновани и моделовани на реалним ситуацијама које су блиске ученику и засноване на његовом искуству, може позитивно утицати на способност за правилно разумевање знака једнакости. Ово истраживање је показало да се употребом реалног контекста процес учења садржаја аритметике и алгебре може унапредити и тако створити снажна основа за правилну изградњу не само појма знака једнакости већ и многих других математичких појмова. Осим тога, моделовање реалних ситуација омогућава ученицима да постепено развијају своја математичка знања, при чему се постепено прелази на формалну математичку симболику и избегавају потенцијални проблеми или неразумевања у садржајима алгебре.*

***Кључне речи**: моделовање реалних ситуација, знак једнакости, рана алгебра, реалистично математичко образовање, ученичка постигнућа*