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ON THE PROBLEM OF KAKEYA

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Abstract

Let $p(z)$ be a complex polynomial of degree n, having k of its zeros in the unit disc. We prove that at least one zero of $p^{(k-1)}(z)$ lies in the disc $|z| \leq \frac{2(n-k+1)}{\ln 2}$.
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Let $p(z)$ be a complex polynomial of degree n, and suppose that some disc D contains k of its zeros. Then we can search for another disc D_1 that contains m zeros of some derivative of $p(z)$, where D_1 depends only on disc D and parameters n, k and m. It is also of special interest to find the best possible (i.e. smallest) disc D_1. This problem is known as the problem of Kakeya, and in general has not been solved yet. For example, in the case of the first derivative, if $k = 2$ and $m = 1$, then the best possible option for D_1 is given by the well-known Alexander–Kakeya theorem [1]. In the present paper we discuss the case of $(k-1)$-th derivative of $p(z)$, for $k > 2$, and $m = 1$. In other words we want to locate a single zero of $(k-1)$-th derivative of $p(z)$, provided that k zeros of $p(z)$ lie in some disc D.

We shall consider a normalized case where D is a closed unit disc. A special case when $p(z) = (z^k - 1)q(z)$ (i.e. when these k zeros are k-th roots of unity) has already been studied in [2].

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Now, let $z_1, z_2, \ldots, z_k$ be n pairwise distinct complex numbers. We shall assume that they lie in the closed unit disc. Let $a_1, a_2, \ldots, a_k$ be solution of the following system of k linear equations:

$$a_1 z_1^m + a_2 z_2^m + \cdots + a_k z_k^m = \delta_{m,k-1},$$

for $0 \leq m \leq k-1$ (where $\delta_{i,j}$ is the Kronecker δ symbol).

This solution exists, since z_i are pairwise distinct. For such a_i and z_i let us define S_m with

$$S_m = a_1 z_1^m + a_2 z_2^m + \cdots + a_k z_k^m$$

for any non-negative integer m. Our goal now is to estimate $|S_m|$. Let

$$q(z) = (z - z_1)(z - z_2) \cdots (z - z_k) = \sum_{i=0}^{k} c_i z^i.$$

Then

$$z_1^t q(z_1) + z_2^t q(z_2) + \cdots + z_k^t q(z_k) = 0$$

obviously holds for any non-negative integer t.

The previous equality can be written in the form

$$S_{k+t} + \quad c_{k-1} S_{k-1+t} + \cdots + c_0 S_t = 0$$

which gives us a recurrence formula

$$S_m = -(c_{k-1} S_{m-1} + \cdots + c_0 S_{m-k}). \tag{1}$$

Now we want to (inductively) show that

$$|S_m| \leq \alpha^{m-k+1}, \tag{2}$$

where $\alpha = \dfrac{1}{\sqrt[k]{2} - 1} < \dfrac{k}{\ln 2}$. For such α holds $(1+\alpha)^k - 2\alpha^k = 0$.

Obviously (2) holds for $0 \leq m \leq k-1$, hence we can assume $m \geq k$. Then from (1) and Viete's rules follows that

$$\begin{aligned}
|S_m| \leq & |c_{k-1}||S_{m-1}| + |c_{k-2}||S_{m-2}| + \cdots + |c_0||S_{m-k}| \\
\leq & \binom{k}{1}|S_{m-1}| + \binom{k}{2}|S_{m-2}| + \cdots + \binom{k}{k}|S_{m-k}| \\
\leq & \binom{k}{1}\alpha^{m-k} + \binom{k}{2}\alpha^{m-k-1} + \cdots + \binom{k}{k}\alpha^{m-2k+1} \\
= & \quad ((1+\alpha)^k - \alpha^k)\alpha^{m-2k+1}.
\end{aligned}$$

On the other hand, we have that $((1+\alpha)^k - \alpha^k)\alpha^{m-2k+1} = \alpha^{m-k+1}$ which completes our induction.

Now let $a(z) = \sum_{k=0}^{n} a_k z^k$ and $b(z) = \sum_{k=0}^{m} b_k z^k$ be two complex polynomials of degree n and m, $m \le n$. If $m < n$, then we shall assume that $b_{m+1} = b_{m+2} = \cdots = b_n = 0$.

For these polynomials we can define operator $A(a,b) = \sum_{k=0}^{n} (-1)^k \frac{a_k b_{n-k}}{\binom{n}{k}}$. If $n = m$ and $A(a,b) = 0$, then a and b are said to be apolar polynomials. For apolar polynomials holds the classical theorem due to Grace:

Theorem 1 (Theorem of Grace)**.** *If all zeros of a polynomial $a(z)$ are contained in some circular region R, then at least one zero of $b(z)$ is contained in R, provided that $a(z)$ and $b(z)$ are apolar.*

We recall that circular region is (open or closed) disc or half-plane or their exterior. Some generalization of the theorem of Grace can be found in [1,3–5]. In the case of $m \le n$ and $A(a,b) = 0$ these polynomials are sometimes called weakly apolar. The following theorem is a kind of generalization of the theorem of Grace.

Theorem 2. *Let $a(z) = \sum_{k=0}^{n} a_k z^k$ and $b(z) = \sum_{k=0}^{m} b_k z^k$ be two complex polynomials of degree n and m, $m \le n$. If $A(a,b) = 0$, then polynomials $a^{(n-m)}(z)$ and $b(z)$ are apolar.*

The above theorem has been proved in [5], and follows from the equality

$$A(a,b) = \frac{(-1)^{n-m}}{n(n-1)\cdots(n-m+1)} A(a^{(n-m)}, b).$$

Suppose now that $z_1, z_2, \ldots, z_k$ are pairwise distinct zeros of a complex polynomial $p(z)$ of degree n. Then $A(p(z), a_i(z-z_i)^n) = 0$, for all i and for any a_i. That implies that $A(p, \sum_{i=0}^{k} a_i(z-z_i)^n) = 0$, i.e. these two polynomials are weakly apolar. Now, due to our preliminary considerations, we can choose a_i such that

$$a_1 z_1^m + a_2 z_2^m + \cdots + a_k z_k^m = \delta_{m,k-1},$$

for $0 \le m \le k-1$ which means that

$$t(z) = \sum_{i=1}^{k} a_i(z-z_i)^n = \sum_{i=k-1}^{n} \binom{n}{i} S_i (-1)^i z^{n-i} = \sum_{i=0}^{n-k+1} t_i z^i,$$

$t_i = (-1)^{n-i}\binom{n}{i} S_{n-i}$ is a polynomial of degree $n-k+1$. From Theorem 1 we can conclude that $p^{(k-1)}(z)$ and $t(z)$ are apolar. By the theorem of Grace it follows that any disc that contains zeros of $t(z)$ must contain at least one zero of $p^{(k-1)}(z)$. So, our next task is to find a disc that contains all zeros of $t(z)$. In order to do that we shall use the following well-known citerion (see [6]):

Theorem 3. *Let* $a(z) = \sum_{k=0}^{n} a_k z^k$ *be a complex polynomial of degree* n. *Then all its zeros lie in the disc* $|z| \le M$, *where*

$$M = 2 \max_i \left\{ \sqrt[n-i]{\left|\frac{a_i}{a_n}\right|}, \quad i = 0, 1, \ldots, n-1 \right\}.$$

In the case of $t(z)$ it is easy to see that

$$M = 2\left|\frac{t_{n-k}}{t_{n-k+1}}\right| = 2\frac{n-k+1}{k}|S_k| \le 2\frac{n-k+1}{k}\alpha = 2\frac{n-k+1}{k(\sqrt[k]{2}-1)} < \frac{2(n-k+1)}{\ln 2}.$$

Therefore we have proved:

Theorem 4. *Let* $p(z)$ *be a complex polynomial of degree* n, *having* k *of its pairwise distinct zeros in the closed unit disc. Then at least one zero of* $p^{(k-1)}(z)$ *lies in the disc* $|z| < \dfrac{2(n-k+1)}{ln2}$.

In the case when these zeros are not pairwise distinct we can also obtain the above theorem for a disc $|z| \le \dfrac{2(n-k+1)}{\ln 2}$. Namely, if we suppose that these k zeros are not necessarily distinct and the above theorem is not true, that means that $p^{(k-1)}(z)$ has no zero in the disc $|z| \le \dfrac{2(n-k+1)}{\ln 2}$. Then we can "very slightly" shift these zeros inside the unit disc, making them pairwise distinct and keeping also zeros of $p^{(k-1)}(z)$ outside the disc $|z| \le \dfrac{2(n-k+1)}{\ln 2}$. But this situation is in contradiction with Theorem 3. Therefore we have proved:

Theorem 5. *Let* $p(z)$ *be a complex polynomial of degree* n, *having* k *of its zeros in the closed unit disc. Then at least one zero of* $p^{(k-1)}(z)$ *lies in the disc* $|z| \le \dfrac{2(n-k+1)}{\ln 2}$.

If $k = n$ (i.e. if we want to locate a zero of $p^{(n-1)}(z)$ provided that all zeros of $p(z)$ lie in the closed unit disc), then the only zero of $p^{(n-1)}(z)$ lies in the closed unit disc for any polynomial $p(z)$ having all its zeros in it. Since disc $|z| \le \dfrac{2(n-k+1)}{\ln 2}$ contains the closed unit disc that means that our estimation is of no use for $k = n$, but it has a sense for other values of k.

At the end let us note that with the above approach we can locate a zero not only of $p^{(k-1)}(z)$, but also of $p^{(i)}(z)$, for $1 \le i < k-1$. In that case the starting point would be to replace $\delta_{m,k-1}$ with $\delta_{m,i}$ at the beginning of page 652, but we find case $i = k-1$ the most interesting.

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