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Systems of *k* Boolean Inequations

ULFETA MAROVAC

State University of Novi Pazar, Department of Mathematics,
Vuka Karadžića bb, 36300 Novi Pazar, Serbia
E-mail: umarovac@np.ac.rs

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The problem of solving generalized systems of Boolean equations is open. One part of this problem is to solve the system of k Boolean inequations. In this paper we give all the solutions of the system of k Boolean inequations.

Boolean equations have an important role in intelligence problems, artificial intelligence, electrical engineering and mathematical programming. Their role in propositional logic and theoretical computer science is well-known. Certain problems of optimization in databases reduce to the solution of system consisting Boolean equations and Boolean inequations. Generalized systems of Boolean equations are built using conjunctions and disjunctions of Boolean equations and Boolean inequations. The methods for finding the solutions of generalized systems of Boolean equations are very important for the development of the above mentioned fields. The problem of solving generalized system of Boolean equations is only partially resolved. Some results can be found in Rudeanu's books [5, 6]. Banković has recently described all the solutions of Boolean inequation, the system of Boolean equation and inequation and the system of two Boolean inequations in [2, 3, 4].

Let $(B, \cap, \cup, ', 0, 1)$ be a Boolean algebra and n be a natural number.

Definition 1. *Let $x \in B$. Then*

$$x^1 = x, \qquad x^0 = x'.$$

If $X = (x_1, \ldots, x_n) \in B^n$ *and* $A = (a_1, \ldots, a_n) \in \{0,1\}^n$ *then*

$$X^A = x_1^{a_1} \cap \cdots \cap x_n^{a_n}.$$

In the sequel $\cap$ *will be omitted.*

Let $u + v = u'v \cup uv'$, *where* $u, v \in B$. *One can prove that* $u = v \Leftrightarrow u + v = 0$. *For the following theorems, see e.g. Rudeanu [5].*

Theorem 1. *The function* $f : B^n \to B$ *is Boolean if and only if it can be written in the canonical disjunctive form*

$$f(X) = \bigcup_A f(A)X^A.$$

Let $f : B^n \to B$ be a Boolean function. The relation $f(X) = 0$ is called a Boolean equation.

Theorem 2. *Let* $x_1, ...x_n, a_c, b_c (C \in \{0,1\}^n \subseteq B^n)$ *be elements of a Boolean algebra* $(B, \cup, \cdot, ', 0, 1)$*; put* $X = (x_1, x_2, ..., x_n)$. *The following relation holds:*

$$(\bigcup_c a_c X^c)(\bigcup_c b_c X^c) = (\bigcup_c a_c b_c X^c).$$

1 GENERALIZED SYSTEMS OF BOOLEAN EQUATIONS

Definition 2. *The generalized systems of Boolean equations (GSBE's for short) over a Boolean algebra are defined recursively as follows:*

- *(i)* *every Boolean equation* $f(X) = 0$ *is a GSBE;*
- *(ii)* *the negation, logical conjunction and logical disjunction of any GSBE's is a GSBE;*
- *(iii)* *every GSBE is obtained by applying rules (i) and (ii) finitely many times.*

The problem of solving GSBE's reduces to a particular case of it.

Definition 3. *An elementary GSBE is either a Boolean equation* $f(X) = 0$ *or the system of the form*

$$f_1(X) \neq 0 \ \wedge \cdots \wedge \ f_k(X) \neq 0 \qquad (1)$$

or of the form

$$g(X) = 0 \wedge f_1(X) \neq 0 \ \wedge \cdots \wedge \ f_k(X) \neq 0. \tag{2}$$

If k=1 we will say that the GSBE is atomic. An atomic GSBE of the form $f(X) \neq 0$ *will be called a Boolean inequation.*

Theorem 3. *The set of solutions of any GSBE is the union of the sets of solutions of several elementary GSBE's.*

The previous two definitions, Theorem 3, and more on generalized systems of Boolean equations can be found e.g. in Rudeanu [6]. In this paper we describe all solutions of the GSBE of the form (1).

2 BOOLEAN EQUATIONS

To solve a Boolean equation $f(X) = 0$ means to determine all such $X \in B^n$ that $f(X) = 0$ holds i.e. to determine the set $S = \{X | f(X) = 0 \wedge X \in B^n\}$.

Theorem 4. *(Theorem 2.3 in [5]) Let* $f : B^n \to B$ *be a Boolean function. The equation* $f(X) = 0$ *has a solution if and only if* $\prod_A f(A) = 0$.

Let $T = (t_1, \ldots, t_n)$.

Definition 4. *Let* $f, F_1, \ldots, F_n : B^n \to B$ *be Boolean functions and* $F = (F_1, \ldots, F_n)$*. The formula*

$$X = F(T),$$

or in scalar form

$$x_i = F_i(t_1, \ldots, t_n), \quad (i = 1, \ldots, n)$$

expresses a general solution of the Boolean equation $f(X) = 0$ *if and only if, for every* $X \in B^n$

$$f(X) = 0 \Leftrightarrow (\exists T) X = F(T).$$

Definition 5. *Let* $f, F_1, \ldots, F_n : B^n \times B^m \to B$ *be Boolean functions and* $F = (F_1, \ldots, F_n)$*. The formula*

$$X = F(T, Y),$$

or in scalar form

$$x_i = F_i(t_1, \dots, t_n, Y), \quad (i = 1, \dots, n)$$

expresses a general solution of the Boolean equation $f(X,Y) = 0$ *by* X *if and only if, for every* $X \in B^n$ *and every* $Y \in B^m$*,*

$$f(X,Y) = 0 \Leftrightarrow (\exists S \in B^n) f(S,Y) = 0 \wedge (\exists T \in B^n) X = F(T,Y).$$

In accordance with Theorem 4 the previous formula can be written as

$$f(X,Y) = 0 \Leftrightarrow \prod_A f(A,Y) = 0 \wedge (\exists T \in B^n) X = F(T,Y).$$

Lemma 1. *(Lemma 2.2 in [5]). Suppose that the equation*

$$ax \cup bx' = 0$$

has a solution $(ab = 0)$*. Then*

$$ax \cup bx' = 0 \Leftrightarrow (\exists t)(x = a't \cup bt') \tag{3}$$

$$ax \cup bx' = 0 \Leftrightarrow b \le x \le a' \tag{4}$$

for all $x \in B$*.*

Theorem 5. *(Theorem 2.11 in [5]) Let* P *be a particular solution of the Boolean equation* $f(X) = 0$*. Then the formula*

$$X = Pf(T) \cup Tf'(T)$$

expresses the reproductive general solution of $f(X) = 0$*.*

Theorem 6. *(Banković [1]) Let* $f : B^n \to B$ *be a Boolean function. If* $f(X) = 0$ *is consistent then , for every* $X \in B^n$

$$\begin{aligned} f(X) = 0 \Leftrightarrow (\exists T)X \quad &= \quad \bigcup_{i=0}^{k} (f'(A_i)A_i \cup f(A_i)f'(A_{i_1})A_{i_1} \cup f(A_i)f(A_{i_1})f'(A_{i_2})A_{i_2} \\ &\cup \cdots \cup f(A_i)f(A_{i_1})f(A_{i_2}) \cdots f(A_{i_{k-1}})f'(A_{i_k})A_{i_k})T^{A_i} \end{aligned}$$

where, for every $i \in \{0, 1, \dots, k\}$*,* $A_i, A_{i_1}, \dots, A_{i_k}$ *is a permutation of* $\{0,1\}^n$*.*

3 BOOLEAN INEQUATIONS

We shall use the following obvious equivalence

$$f(X) \neq 0 \Leftrightarrow (\exists p)(p \neq 0 \wedge f(X) = p). \tag{5}$$

Let $f : B^n \to B$ be a Boolean function. The relation

$$f(X) \neq 0$$

is called a Boolean inequation.

To solve a Boolean inequation $f(X) \neq 0$ means to determine all such $X \in B^n$ that $f(X) \neq 0$ holds.

Theorem 7. *(Remark 10.5 in [5]) Let $f : B^n \to B$ be a Boolean function. The inequation $f(X) \neq 0$ has a solution if and only if $\bigcup_A f(A) \neq 0$.*

Theorem 8. *(Theorem 5 in [3]) Let $f : B^n \to B$ be a Boolean function. Then*

$$f(X) \neq 0 \Leftrightarrow (\exists p)(p \neq 0 \wedge \bigcup_A ((f(A) + p)X^A) = 0).$$

Theorem 9. *(Theorem 6 in [3]) Let $f : B^n \to B$ be a Boolean function. Suppose that the inequation $f(X) \neq 0$ has a solution. Let $X = \Phi(T, p)$ expresses the general solution of the equation*

$$\bigcup_A ((f(A) + p)X^A) = 0.$$

Then, for every $X \in B^n$,

$$f(X) \neq 0 \Leftrightarrow (\exists p)(\exists T)(p \neq 0 \wedge \prod_A f(A) \leq p \leq \bigcup_A f(A) \wedge X = \Phi(T, p)).$$

4 SYSTEMS OF K BOOLEAN INEQUATIONS

In this section we shall consider the system

$$f_1(X) \neq 0 \ \wedge \cdots \wedge \ f_k(X) \neq 0$$

where $f_1, \ldots, f_k : B^n \to B$ are Boolean functions.

Theorem 10. *(Theorem 7 in [4]) Let P be a particular solution of the system $f_1(X) \neq 0 \wedge \cdots \wedge f_k(X) \neq 0$, where $f_1, \ldots, f_k : B^n \to B$ are Boolean functions. Then, for every $T \in B^n$,*

$$X = (f_1(T) \cdots f_k(T))' P \cup f_1(T) \cdots f_k(T) T \Rightarrow f_1(X) \neq 0 \wedge \cdots \wedge f_k(X) \neq 0.$$

Lemma 2. *Let $f_1, \ldots, f_k : B^n \to B$ be Boolean functions. Then*

$$f_1(X) \neq 0 \ \wedge \cdots \wedge \ f_k(X) \neq 0 \Leftrightarrow \tag{6}$$

$(\exists p_1) \cdots (\exists p_k)(p_1 \neq 0 \wedge \cdots \wedge p_k \neq 0 \wedge ((f_1(X) + p_1) \cup \cdots \cup (f_k(X) + p_k)) = 0)$.

Proof. By using (5) we get

$$\begin{aligned}
& f_1(X) \neq 0 \ \wedge \ \cdots \ \wedge \ f_k(X) \neq 0 \\
\Leftrightarrow\ & (\exists p_1)(p_1 \neq 0 \wedge f_1(X) = p_1) \wedge \cdots \wedge (\exists p_k)(p_k \neq 0 \wedge f_k(X) = p_k) \\
\Leftrightarrow\ & (\exists p_1) \cdots (\exists p_k)(p_1 \neq 0 \wedge f_1(X) = p_1 \wedge \cdots \wedge p_k \neq 0 \wedge f_k(X) = p_k) \\
\Leftrightarrow\ & (\exists p_1) \cdots (\exists p_k)(p_1 \neq 0 \wedge \cdots \wedge p_k \neq 0 \wedge f_1(X) + p_1 = 0 \wedge \cdots \wedge f_k(X) + p_k = 0) \\
\Leftrightarrow\ & (\exists p_1) \cdots (\exists p_k)(p_1 \neq 0 \wedge \cdots \wedge p_k \neq 0 \wedge (f_1(X) + p_1) \cup \cdots \cup (f_k(X) + p_k) = 0\big).
\end{aligned}$$

Lemma 3. *Let $f_1, \ldots, f_k : B^n \to B$ be Boolean functions and $p_1, \ldots, p_k \in B$. Then*

$$\begin{aligned}
&\prod_A ((f_1(A) + p_1) \cup \cdots \cup (f_k(A) + p_k)) = \\
&\bigcup_{(c_1,\ldots,c_k)\in\{0,1\}^k} {p_1}^{c_1} \cdots {p_k}^{c_k} \prod_A ({f_1}^{c_1'}(A) \cup \cdots \cup {f_k}^{c_k'}(A)).
\end{aligned} \tag{7}$$

Proof. Let F be the Boolean function defined by

$$F(p_1, \ldots, p_k) = \prod_A ((f_1(A) + p_1) \cup \cdots \cup (f_k(A) + p_k)).$$

Then, by Theorem 1,

$$\begin{aligned}
F(p_1, \ldots, p_k) &= \bigcup_{(c_1,\ldots,c_k)\in\{0,1\}^k} {p_1}^{c_1} \cdots {p_k}^{c_k} F(c_1, \ldots, c_k) \\
&= \bigcup_{(c_1,\ldots,c_k)\in\{0,1\}^k} {p_1}^{c_1} \cdots {p_k}^{c_k} \prod_A ((f_1(A) + c_1) \cup \cdots \cup (f_k(A) + c_k)).
\end{aligned}$$

Since $f_i(A) + c_i = f_i(A) = f_i^{c_i'}(A)$ for $c_i = 0$ and $f_i(A) + c_i = f_i'(A) = f_i^{c_i'}(A)$ for $c_i = 1$, for every $i \in (1, \dots, k)$ it follows that

$$F(p_1, \dots, p_k) = \bigcup_{(c_1,\dots,c_k)\in\{0,1\}^k} p_1^{c_1} \cdots p_k^{c_k} \prod_A (f_1^{c_1'}(A) \cup \cdots \cup f_k^{c_k'}(A)).$$

Let $C_i = (c_1, \dots, c_i)$.

Lemma 4. *Let $f_1, \dots, f_k : B^n \to B$ be Boolean functions. Then the equation*

$$\prod_A ((f_1(A) + p_1) \cup \cdots \cup (f_k(A) + p_k)) = 0 \tag{8}$$

in $p_1, \dots, p_k$ has a solution.

Proof. By using Lemma 3 the equation (8) can be written as

$$\bigcup_{C_k\in\{0,1\}^k} p_1^{c_1} \cdots p_k^{c_k} \prod_A (f_1^{c_1'}(A) \cup \cdots \cup f_k^{c_k'}(A)) = 0.$$

This equation has a solution if and only if

$$\prod_{C_k\in\{0,1\}^k} \prod_A (f_1^{c_1'}(A) \cup \cdots \cup f_k^{c_k'}(A)) = 0$$

by Theorem 4. Let $G(A) = (f_1(A), \dots, f_k(A))$. Since $\bigcup_C Y^C = 1$ for every natural number n and every $Y \in B^n$, it follows that

$$\begin{aligned} \textstyle\bigcup_C (G(A))^C &= \textstyle\bigcup_{C_k\in\{0,1\}^k} (f_1(A), \dots, f_k(A))^{C_k} \\ &= \textstyle\bigcup_{C_k\in\{0,1\}^k} (f_1(A)^{c_1} \cdots f_k(A)^{c_1}) = 1. \end{aligned}$$

If we apply the negation on the last equality we get

$$\prod_{C_k\in\{0,1\}^k} (f_1^{c_1'}(A) \cup \cdots \cup f_k^{c_k'}(A)) = 0. \tag{9}$$

Then

$$\prod_A \prod_{C_k\in\{0,1\}^k} (f_1^{c_1'}(A) \cup \cdots \cup f_k^{c_k'}(A)) = \prod_{C_k\in\{0,1\}^k} \prod_A (f_1^{c_1'}(A) \cup \cdots \cup f_k^{c_k'}(A)) = 0$$

because of (9). Therefore the equation (8) has a solution.

Lemma 5. *Let* $f_1, \ldots, f_k : B^n \to B$ *be Boolean functions and* $p_1, \ldots, p_k \in B$ *then*

$$\prod_A((f_1(A)+p_1)\cup\cdots\cup(f_k(A)+p_k)) = 0 \Leftrightarrow$$
$$\bigcup_{C_{k-1}\in\{0,1\}^{k-1}} {p_1}^{c_1}\cdots {p_{k-1}}^{c_{k-1}} \prod_A(f_1^{c_1'}(A)\cup\cdots\cup f_{k-1}^{c_{k-1}'}(A)\cup f_k(A))$$
$$\le p_k \le \bigcup_{C_{k-1}\in\{0,1\}^{k-1}} p_1^{c_1}\cdots p_{k-1}^{c_{k-1}} \bigcup_A(f_1^{c_1}(A)\cdots f_{k-1}^{c_{k-1}}(A) f_k(A))$$
$$\wedge \quad \prod_A((f_1(A)+p_1)\cup\cdots\cup(f_{k-1}(A)+p_{k-1})) = 0.$$

Proof. By using (7) we have

$$\begin{aligned}
&\prod_A((f_1(A)+p_1)\cup\cdots\cup(f_k(A)+p_k))\\
&= \bigcup_{C_k\in\{0,1\}^k} {p_1}^{c_1}\cdots {p_k}^{c_k} \prod_A(f_1^{c_1'}(A)\cup\cdots\cup f_k^{c_k'}(A))\\
&= p_k(\bigcup_{C_{k-1}\in\{0,1\}^{k-1}} {p_1}^{c_1}\cdots {p_{k-1}}^{c_{k-1}} \prod_A(f_1^{c_1'}(A)\cup\cdots\cup f_{k-1}^{c_{k-1}'}(A)\cup f_k'(A)))\\
&\cup p_k'(\bigcup_{C_{k-1}\in\{0,1\}^{k-1}} {p_1}^{c_1}\cdots {p_{k-1}}^{c_{k-1}} \prod_A(f_1^{c_1'}(A)\cup\cdots\cup f_{k-1}^{c_{k-1}'}(A)\cup f_k(A))) = 0.
\end{aligned}$$

Let us introduce the following notation

$$\begin{aligned}
a &= (\bigcup_{C_{k-1}\in\{0,1\}^{k-1}} {p_1}^{c_1}\cdots {p_{k-1}}^{c_{k-1}} \prod_A(f_1^{c_1'}(A)\cup\cdots\cup f_{k-1}^{c_{k-1}'}(A)\cup f_k'(A))))\\
b &= (\bigcup_{C_{k-1}\in\{0,1\}^{k-1}} {p_1}^{c_1}\cdots {p_{k-1}}^{c_{k-1}} \prod_A(f_1^{c_1'}(A)\cup\cdots\cup f_{k-1}^{c_{k-1}'}(A)\cup f_k(A))).
\end{aligned}$$

Then by applying Theorem 2 where $n = k-1$ and $X = (p_1, p_2, \ldots, p_{k-1})$ we get

$$\begin{aligned}
ab = \; &\bigcup_{C_{k-1}\in\{0,1\}^{k-1}} {p_1}^{c_1}\cdots {p_{k-1}}^{c_{k-1}}(\prod_A(f_1^{c_1'}(A)\cup\cdots\cup f_{k-1}^{c_{k-1}'}(A)\cup f_k'(A)))\\
&(\prod_A(f_1^{c_1'}(A)\cup\cdots\cup f_{k-1}^{c_{k-1}'}(A)\cup f_k(A))).
\end{aligned}$$

By using the equality $(x\cup y)(x\cup y') = x$, we get

$$\begin{aligned}
(f_1^{c_1'}(A)\cup\cdots \cup\; &f_{k-1}^{c_{k-1}'}(A)\cup f_k(A))(f_1^{c_1'}(A)\cup\cdots\cup f_{k-1}^{c_{k-1}'}(A)\cup f_k'(A))\\
&= f_1^{c_1'}(A)\cup\cdots\cup f_{k-1}^{c_{k-1}'}(A).
\end{aligned}$$

Thus

$$ab = \bigcup_{C_{k-1}\in\{0,1\}^{k-1}} {p_1}^{c_1}\cdots {p_{k-1}}^{c_{k-1}}(\prod_A (f_1^{c_1'}(A)\cup\cdots\cup f_{k-1}^{c_{k-1}'}(A))).$$

In accordance with Lemma 3 we have

$$ab = \quad \prod_A((f_1(A)+p_1)\cup\cdots\cup(f_{k-1}(A)+p_{k-1})).$$

The equation $ap_k\cup bp_k' = 0$ has a solution if and only if $ab = 0$, by Lemma 1. The equality $ab = 0$ can be written as

$$\prod_A((f_1(A)+p_1)\cup\ldots\cup(f_{k-1}(A)+p_{k-1})) = 0.$$

This equation has a solution, by Lemma 4. In accordance with Lemma 1, the equation $ap_k\cup bp_k' = 0$ is equivalent to $b\le p_k\le a'$, i.e.

$$\bigcup_{C_{k-1}\in\{0,1\}^{k-1}} {p_1}^{c_1}\cdots {p_{k-1}}^{c_{k-1}}\prod_A (f_1^{c_1'}(A)\cup\cdots\cup f_{k-1}^{c_{k-1}'}(A)\cup f_k(A))$$

$$\le p_k\le \bigcup_{C_{k-1}\in\{0,1\}^{k-1}} {p_1}^{c_1}\cdots p_{k-1}^{c_{k-1}}\bigcup_A (f_1^{c_1}(A)\cdots f_{k-1}^{c_{k-1}}(A)f_k(A)).$$

Theorem 11. *Let $f_1,\ldots, f_k : B^n\to B$ be Boolean function. Then*

$$f_1(X)\ne 0\wedge\cdots\wedge f_k(X)\ne 0\Leftrightarrow$$

$$(\exists p_1)\cdots(\exists p_k)(\exists T)(p_1\ne 0\wedge\cdots\wedge p_k\ne 0\wedge X = \Phi(p_1,\ldots,p_k,T)$$

$$\wedge\ \prod_A f_1(A)\le p_1\le\bigcup_A f_1(A)$$

$$\wedge\ p_1\prod_A(f_1'(A)\cup f_2(A))\cup p_1'\prod_A(f_1(A)\cup f_2(A))$$

$$\le p_2\le p_1\bigcup_A(f_1(A)f_2(A))\cup p_1'\bigcup_A(f_1'(A)f_2(A))$$

$$\cdots$$

$$\bigcup_{C_{k-1}\in\{0,1\}^{k-1}} {p_1}^{c_1}\cdots {p_{k-1}}^{c_{k-1}}\prod_A(f_1^{c_1'}(A)\cup\cdots\cup f_{k-1}^{c_{k-1}'}(A)\cup f_k(A))$$

$$\le p_k\le\bigcup_{C_{k-1}\in\{0,1\}^{k-1}} {p_1}^{c_1}\cdots p_{k-1}^{c_{k-1}}\bigcup_A(f_1^{c_1}(A)\cdots f_{k-1}^{c_{k-1}}(A)f_k(A))),$$

where $X = \Phi(p_1, \ldots, p_k, T)$ *expresses the general solution of the equation*

$$(f_1(X) + p_1) \cup \ldots \cup (f_k(X) + p_k) = 0.$$

Proof. By Lemma 2 equivalence (6) holds. Let $X = \Phi(p_1, \ldots, p_k, T)$ be a general solution of the equation

$$(f_1(X) + p_1) \cup \cdots \cup (f_k(X) + p_k) = 0. \tag{10}$$

Then, by Definition 5,

$$(f_1(X) + p_1) \cup \cdots \cup (f_k(X) + p_k) = 0 \Leftrightarrow \tag{11}$$

$$\prod_A ((f_1(A) + p_1) \cup \cdots \cup (f_k(A) + p_k)) = 0 \;\wedge\; (\exists T) X = \Phi(p_1, \ldots, p_k, T).$$

The condition

$$\prod_A ((f_1(A) + p_1) \cup \cdots \cup (f_k(A) + p_k)) = 0$$

is an equation in $p_1, \ldots p_k$. This equation is consistent by Lemma 4 and according to Lemma 5 it is equivalent to the following conjunction

$$\bigcup\nolimits_{C_{k-1} \in \{0,1\}^{k-1}} {p_1}^{c_1} \cdots {p_{k-1}}^{c_{k-1}} \prod\nolimits_A (f_1^{c_1'}(A) \cup \cdots \cup f_{k-1}^{c_{k-1}'}(A) \cup f_k(A))$$

$$\leq p_k \leq \bigcup\nolimits_{C_{k-1} \in \{0,1\}^{k-1}} {p_1}^{c_1} \cdots p_{k-1}^{c_{k-1}} \bigcup\nolimits_A (f_1^{c_1}(A) \cdots f_{k-1}^{c_{k-1}}(A) f_k(A))$$

$$\wedge \; \prod\nolimits_A ((f_1(A) + p_1) \cup \cdots \cup (f_{k-1}(A) + p_{k-1})) = 0.$$

Similarly, according to Lemma 5 it follows that

$$\prod\nolimits_A ((f_1(A) + p_1) \cup \cdots \cup (f_{k-1}(A) + p_{k-1})) = 0 \Leftrightarrow$$

$$\bigcup\nolimits_{C_{k-2} \in \{0,1\}^{k-2}} {p_1}^{c_1} \cdots {p_{k-2}}^{c_{k-2}} \prod\nolimits_A (f_1^{c_1'}(A) \cup \cdots \cup f_{k-1}^{c_{k-2}'}(A) \cup f_{k-1}(A))$$

$$\leq p_{k-1} \leq \bigcup\nolimits_{C_{k-2} \in \{0,1\}^{k-2}} {p_1}^{c_1} \cdots p_{k-2}^{c_{k-2}} \bigcup\nolimits_A (f_1^{c_1}(A) \cdots f_{k-1}^{c_{k-2}}(A) f_{k-1}(A))$$

$$\wedge \; \prod\nolimits_A ((f_1(A) + p_1) \cup \cdots \cup (f_{k-2}(A) + p_{k-2})) = 0.$$

By applying $(k-1)$ times Lemma 5 we get $\prod_A(f_1(A)+p_1)=0$, which can be written as $\prod_A f_1'(A)p_1 \cup \prod_A f_1(A)p_1' = 0$. This equation in p_1 has a solution because $\prod_A f_1'(A)\prod_A f_1(A)=0$ and the solutions are determined by

$$\prod_A f_1(A) \le p_1 \le \bigcup_A f_1(A).$$

From (6), (11) and the previous conditions for $p_1, \ldots, p_k$ we get Theorem 11.

Let $m = 2^n - 1$. According to Theorem 6 the general solution of the equation (10) can be obtained as follows:

$$\Phi(p_1,\ldots,p_k,T)) = \bigcup_{i=0}^{m}(((f_1(A_i)+p_1)\cup\cdots\cup(f_k(A_i)+p_k))'A_i \tag{12}$$

$$\cup((f_1(A_i)+p_1)\cup\cdots\cup(f_k(A_i)+p_k))((f_1(A_{i_1})+p_1)\cup\cdots\cup(f_k(A_{i_1})+p_k))'A_{i_1}$$

$$\cup((f_1(A_i)+p_1)\cup\cdots\cup(f_k(A_i)+p_k))((f_1(A_{i_1})+p_1)\cup\cdots\cup(f_k(A_{i_1})+p_k))\cdots$$

$$((f_1(A_{i_{m-1}})+p_1)\cdots\cup(f_k(A_{i_{m-1}})+p_k))'A_{i_{m-1}}\cup((f_1(A_i)+p_1)\cup\cdots\cup$$

$$(f_k(A_i)+p_k))\cdots((f_1(A_{i_m})+p_1)\cup\cdots\cup(f_k(A_{i_m})+p_k))'A_{i_m})T^{A_i}$$

where for every $i \in \{0,\ldots,m\}$, $(A_i, A_{i_1}, \ldots, A_{i_m})$ is a permutation of $\{0,1\}^n$.

Example 1. *Let $a, b, c, d, e, f \in B$. Solve the system*

$$ax \cup bx' \ne 0 \wedge cx \cup dx' \ne 0 \wedge ex \cup fx' \ne 0.$$

By applying Theorem 11 and (12) where $k = 3$ and $n = 1$ we get

$$ax \cup bx' \ne 0 \wedge cx \cup dx' \ne 0 \wedge ex \cup fx' \ne 0 \Leftrightarrow$$

$$(\exists p)(\exists q)(\exists r)(\exists t)\big(p \ne 0 \wedge q \ne 0 \wedge r \ne 0 \wedge ab \le p \le a \cup b$$

$$\wedge\, p\,(a' \cup c)(b' \cup d) \cup p'(a \cup c)(b \cup d) \le q \le p\,(ac \cup bd) \cup p'(a'c \cup b'd)$$

$$\wedge\quad pq(a' \cup c' \cup e)(b' \cup d' \cup f) \cup pq'(a' \cup c \cup e)(b' \cup d \cup f)\cup$$

$$p'q(a \cup c' \cup e)(b \cup d' \cup f) \cup p'q'(a \cup c \cup e)(b \cup d \cup f)$$

$$\le r \le pq(ace \cup bdf) \cup pq'(ac'e \cup bd'f) \cup p'q(a'ce \cup b'df) \cup p'q'(a'c'e \cup b'd'f)$$

$$\wedge\; x = ((a+p) \cup (c+q) \cup (e+r))'t \cup ((b+p) \cup (d+q) \cup (f+r))t'\big).$$

Example 2. *Let $B = \{0, 1, m, l, k, m', l', k'\}$. Solve the system*

$$m'x' \neq 0 \wedge m'x \neq 0 \wedge mx \cup lx' \neq 0.$$

In accordance with Example 1 we have $a = 0, b = m', c = m', d = 0, e = m, f = l$ and

$$m'x' \neq 0 \wedge m'x \neq 0 \wedge mx \cup lx' \neq 0 \Leftrightarrow$$

$$(\exists p)(\exists q)(\exists r)(\exists t)\big(p \neq 0 \wedge q \neq 0 \wedge r \neq 0 \wedge 0 \leq p \leq m' \wedge mp \cup m'p' \leq q \leq m'p'$$

$$\wedge\ pq \cup k'pq' \cup mp'q \cup m'p'q' \leq r \leq lpq' \cup mp'q'$$

$$\wedge\ x = (p \cup (m' + q) \cup (m + r))'t \cup ((m' + p) \cup q \cup (l + r))t').$$

The conditions $p \neq 0$ and $0 \leq p \leq m'$ give $p \in \{l, k, m'\}$.

If $p = l$, it follows that $k \leq q \leq k$ and $l \leq r \leq k'$ i.e. $q = k$ and $r \in \{l, k'\}$.
If $p = l$, $q = k$ and $r = l$, we get $x = kt \cup kt' = k$.
If $p = l$, $q = k$ and $r = k'$, we get $x = l't \cup l't' = l'$.
If $p = k$, it follows that $l \leq q \leq l$ and $0 \leq r \leq m$. Then we have $q = l, r = m$ and $x = k't \cup k't' = k'$.
If $p = m'$, we get $q \neq 0$ and $q = 0$. Since conditions $q \neq 0$ and $q = 0$ are contradictory, in this case we do not have any solution x.

The set of all the solutions of the system $m'x' \neq 0 \wedge m'x \neq 0 \wedge mx \cup lx' \neq 0$ in $B = \{0, k, l, m, k', l', m', 1\}$ is $\{k, l', k'\}$.

Example 3. *Let $B = \{0, 1, m, l, k, m', l', k'\}$. Solve the system*

$$lx' \neq 0 \wedge k'x \neq 0 \wedge lx \cup mx' \neq 0. \tag{13}$$

In accordance with Example 1 we have $a = 0, b = l, c = k', d = 0, e = l, f = m$ and

$$lx' \neq 0 \wedge k'x \neq 0 \wedge lx \cup mx' \neq 0 \Leftrightarrow$$

$$(\exists p)(\exists q)(\exists r)(\exists t)\big(p \neq 0 \wedge q \neq 0 \wedge r \neq 0 \wedge 0 \leq p \leq l \wedge l'p \cup lp' \leq q \leq k'p'$$

$$\wedge\ pq \cup l'pq' \cup m'p'q \cup k'p'q' \leq r \leq lp'q \cup mp'q'$$

$$\wedge\ x = (p \cup (k' + q) \cup (l + r))'t \cup ((l + p) \cup q \cup (m + r))t'\big).$$

The conditions $p \neq 0$ *and* $0 \leq p \leq l$ *imply* $p = l$*. The conditions* $q \neq 0$ *and* $0 \leq q \leq m$ *imply* $q = m$*. Consequently* $0 \leq r \leq 0$ *which is opposite to* $r \neq 0$*. Then there is no* r *which satisfy the conditions from Example 1, i.e. the system (13) has no solution in* $B = \{0, 1, m, l, k, m', l', k'\}$*.*

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