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System of Two Boolean Inequations*

DRAGIĆ BANKOVIĆ AND ULFETA MAROVAC

State University of Novi Pazar, Vuka Karadžića bb, 36300 Novi Pazar, Serbia
E-mail: bankovicdragic@gmail.com

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In this paper we consider the system of Boolean ineqations $f(X) \neq 0 \wedge g(X) \neq 0$. We give the formula which determines all the solutions of this system.

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Boolean equations were researched by many authors . The basic facts and various forms of the solutions of Boolean equations can be found in Rudeanu's book [3]. The results published after 1974 were presented in Rudeanu's book [4] (Chapter 6).

Boolean inequations were considered by Schröder [5]. Banković recently described all the solutions of Boolean inequations [2].

The problem of solving system of inequations and/or equations in Boolean algebras is far from being solved in a satisfactory manner. Some results can be found in Chapter 10 of [3] and Chapter 6 of [4]. This paper is a contribution to the solving the mentioned problem.

Let $(B, \cap, \cup, ', 0, 1)$ be a Boolean algebra and n be a natural number.

Definition 1. *Let $x \in B$. Then*

$$x^1 = x, \qquad x^0 = x'.$$

If $X = (x_1, \ldots, x_n) \in B^n$ and $A = (a_1, \ldots, a_n) \in \{0, 1\}^n$ then

$$X^A = x_1^{a_1} \cap \cdots \cap x_n^{a_n}.$$

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In the sequel $\cap$ *will be omitted.*

Let $u + v = u'v \cup uv'$, *where* $u, v \in B$. *One can prove that* $u = v \Leftrightarrow u + v = 0$.

Theorem 1. *(Rudeanu [3], Corollary 1). The function* $f : B^n \to B$ *is Boolean if and only if it can be written in the canonical disjunctive form*

$$f(X) = \bigcup_A f(A)X^A.$$

1 BOOLEAN EQUATIONS

Let $f : B^n \to B$ be a Boolean function. The relation

$$f(X) = 0$$

is called a Boolean equation.

To solve Boolean equation $f(X) = 0$ means to determine all $X \in B^n$ such that $f(X) = 0$ holds i.e. to determine the set $S = \{X | f(X) = 0 \wedge X \in B^n\}$.

Theorem 2. *(Theorem 2.3 in [3]) Let* $f : B^n \to B$ *be a Boolean function. The equation* $f(X) = 0$ *is consistent (has a solution) if and only if* $\prod_A f(A) = 0$.

Let $T = (t_1, \ldots, t_n)$.

Definition 2. *Let* $f, F_1, \ldots, F_n : B^n \to B$ *be Boolean functions and* $F = (F_1, \ldots, F_n)$. *The formulas*

$$X = F(T),$$

or in scalar form

$$x_i = F_i(t_1, \ldots, x_n), \quad (i = 1, \ldots, n)$$

express a general solution of Boolean equation $f(X) = 0$ *if and only if the equation is consistent and for every* $X \in B^n$

$$f(X) = 0 \Leftrightarrow (\exists T)X = F(T).$$

Definition 3. *Let* $f, F_1, \ldots, F_n : B^n \times B^m \to B$ *be Boolean functions and* $F = (F_1, \ldots, F_n)$. *The formula*

$$X = F(T, Y),$$

or in scalar form

$$x_i = F_i(t_1, \ldots, x_n, Y), \quad (i = 1, \ldots, n)$$

expresses a general solution of Boolean equation $f(X, Y) = 0$ *by* X *if and only if , for every* $X \in B^n$ *and every* $Y \in B^m$*,*

$$f(X, Y) = 0 \Leftrightarrow (\exists S \in B^n) f(S, Y) = 0 \wedge (\exists T \in B^n) X = F(T, Y).$$

In accordance with Theorem 2 the previous formula can be written as

$$f(X, Y) = 0 \Leftrightarrow \prod_A f(A, Y) = 0 \wedge (\exists T \in B^n) X = F(T, Y).$$

Lemma 1. *(Lemma 2.2 in [3]). Assume that the equation*

$$cx \cup dx' = 0$$

has a solution $(ab = 0)$*. Then*

$$cx \cup dx' = 0 \Leftrightarrow (\exists t)(x = c't \cup dt')$$

$$cx \cup dx' = 0 \Leftrightarrow b \leq x \leq a'$$

for all $x \in B$*.*

Lemma 2. *(Lemma 2.3. in [3]). Let* $f : B^n \to B$ *be a Boolean function. Then, for every* $u \in B, V, W \in B^n$*,*

$$f(uV \cup u'W) = uf(V) \cup u'f(W).$$

Theorem 3. *(Theorem 2.11 in [3]) Let* P *be a particular solution of the Boolean equation* $f(X) = 0$*. Then the formula*

$$X = Pf(T) \cup Tf'(T)$$

expresses the reproductive general solution of $f(X) = 0$*.*

2 BOOLEAN INEQUATIONS

We shall use the following obvious equivalence

$$f(X) \neq 0 \Leftrightarrow (\exists p)(p \neq 0 \wedge f(X) = p).$$

Let $f : B^n \to B$ be a Boolean function. The relation

$$f(X) \neq 0$$

is called a Boolean inequation.

To solve Boolean a inequation $f(X) \neq 0$ means to determine all $X \in B^n$ such that $f(X) \neq 0$ holds.

Theorem 4. *(Remark 10.5 in [3]) Let $f : B^n \to B$ be a Boolean function. Inequation $f(X) \neq 0$ has a solution if and only if $\bigcup_A f(A) \neq 0$.*

Theorem 5. *(Theorem 6 in [2]) Let $f : B^n \to B$ be a Boolean function. Then*

$$f(X) \neq 0 \Leftrightarrow (\exists p)(p \neq 0 \wedge \bigcup_A ((f(A) + p))X^A = 0).$$

Theorem 6. *(Theorem 5 in [2]) Let $f : B^n \to B$ be Boolean function. Suppose that the inequation $f(X) \neq 0$ has a solution. Let $X = \Phi(T, p)$ express the general solution of the equation*

$$\bigcup_A ((f(A) + p))X^A = 0.$$

Then, for every $X \in B^n$, $f(X) \neq 0 \Leftrightarrow (\exists p)(\exists T)(p \neq 0 \wedge \prod_A f(A) \leq p \leq \bigcup_A f(A) \wedge X = \Phi(T, p))$.

3 SYSTEM OF BOOLEAN INEQUATIONS

In this section we shall consider the system

$$f(X) \neq 0 \;\; \wedge \;\; g(X) \neq 0$$

where $f, g : B^n \to B$ are Boolean functions.

Theorem 7. *Let P be a particular solution of the system $f_1(X) \neq 0 \wedge \cdots \wedge f_k(X) \neq 0$, where $f_1, \ldots, f_k : B^n \to B$ are Boolean functions. Then, for every $T \in B^n$,*

$$X = (f_1(T) \cdots f_k(T))' P \cup f_1(T) \cdots f_k(T) T \Rightarrow f_1(X) \neq 0 \wedge \cdots \wedge f_k(X) \neq 0.$$

Proof. Let $X = (f_1(T)\cdots f_k(T))'P \cup f_1(T)\cdots f_k(T)T$. Then it follows from the hypotheses and Lemma 2 that for every $j \in \{1, \ldots, k\}$

$$\begin{aligned} f_j(X) &= (f_1(T)\cdots f_k(T))' f_j(P) \cup f_1(T)\cdots f_k(T) f_j(T) \\ &= (f_1(T)\cdots f_k(T))' f_j(P) \cup f_1(T)\cdots f_k(T). \end{aligned}$$

If $f_1(T)\cdots f_k(T) \neq 0$ then $f_j(X) \neq 0$. If $f_1(T)\cdots f_k(T) = 0$ then again $f_j(X) = f_j(P) \neq 0$.

Lemma 3. *Let $f, g : B^n \to B$ Boolean functions and $p, q \in B$. Then*

$$\prod_A((f(A)+p)\cup(g(A)+q)) = pq\ \prod_A(f'(A)\cup g'(A))$$
$$\cup pq'\ \prod_A(f'(A)\cup g(A)) \cup p'q\ \prod_A(f(A)\cup g'(A)) \cup p'q'\ \prod_A(f(A)\cup g(A)).$$

Proof. Let us introduce the notation $F(p,q) = \prod_A((f(A)+p)\cup(g(A)+q))$. Since $F(p,q) = F(1,1)pq \cup F(1,0)pq' \cup F(0,1)p'q \cup F(0,0)p'q'$ we get Lemma 3.

Lemma 4. *Let $f, g : B^n \to B$ be Boolean functions. Then the equation $\prod_A((f(A)+p)\cup(g(A)+q)) = 0$, in p and q, has solution.*

Proof. One can prove the equality

$$\prod_A(f'(A)\cup g'(A))\prod_A(f'(A)\cup g(A))\prod_A(f(A)\cup g'(A))\prod_A(f(A)\cup g(A)) = 0.$$

In accordance to Lemma 3 and Theorem 2, the equation $\prod_A((f(A)+p)\cup(g(A)+q)) = 0$ has solution.

Lemma 5. [5] *Let $a, b, c, d \in B$. If $abcd = 0$ then*

$$axy \cup bxy' \cup cx'y \cup dx'y' = 0 \Leftrightarrow cd \le x \le a' \cup b' \ \wedge\ bx \cup dx' \le y \le a'x \cup c'x'.$$

Theorem 8. *Let $f, g : B^n \to B$ be Boolean functions. Then*

$$f(X) \neq 0 \wedge g(X) \neq 0 \Leftrightarrow$$
$$(\exists p)(\exists q)(\exists T)\big(p \neq 0 \wedge q \neq 0 \wedge\ X = \Phi(p,q,T)\ \wedge \prod_A f(A) \le p \le \bigcup_A f(A)\ \wedge$$
$$p\prod_A(f'(A)\cup g(A)) \cup p'\prod_A(f(A)\cup g(A)) \le q \le p\bigcup_A f(A)g(A) \cup p'\bigcup_A f'(A)g(A)\big),$$

where $X = \Phi(p,q,T)$ expresses the general solution of the equation $(f(X)+p)\cup(g(X)+q) = 0$.

Proof. System $f(X) \neq 0 \wedge g(X) \neq 0$ can be written as

$$(\exists p)(\exists q)\big(p \neq 0 \wedge q \neq 0 \wedge f(X) = p \wedge g(X) = q\big)$$

i.e.

$$\begin{aligned} f(X) \neq 0 \wedge g(X) &\neq 0 \Leftrightarrow (\exists p)(\exists q)(p \neq 0 \wedge q \neq 0 \wedge (f(X) \\ &+ p) \cup (g(X) + q) = 0). \end{aligned} \tag{1}$$

Let $X = \Phi(T, p, q)$ be a general solution of $(f(X) + p) \cup (g(X) + q) = 0$ i.e.

$$\begin{aligned} (f(X) + p) \cup (g(X) + q) &= 0 \Leftrightarrow \prod_A ((f(A) + p) \cup (g(A) + q)) \\ &= 0 \wedge (\exists T) X = \Phi(p, q, T), \end{aligned} \tag{2}$$

by Definition 3. The condition $\prod_A((f(A) + p) \cup (g(A) + q)) = 0$ is an equation in p and q. This equation is consistent, by Lemma 4. It can be written as $pq \prod_A (f'(A) \cup g'(A)) \cup pq' \prod_A (f'(A) \cup g(A)) \cup p'q \prod_A (f(A) \cup g'(A)) \cup p'q' \prod_A (f(A) \cup g(A)) = 0$, where, by Lemma 5, we get

$$\prod_A ((f(A) + p) \cup (g(A) + q)) = 0 \Leftrightarrow \prod_A f(A) \leq p \leq \bigcup_A f(A) \wedge \tag{3}$$

$$p \prod_A (f'(A) \cup g(A)) \cup p' \prod_A (f(A) \cup g(A)) \leq q \leq p \bigcup_A f(A) g(A) \cup p' \bigcup_A f'(A) g(A)\big).$$

Using (1), (2) and (3) we get Theorem 8.

Taking $n = 1$ we have

Corollary 1. *Let* $a, b, c, d \in B$ *Then*

$$\begin{gathered} ax \cup bx' \neq 0 \wedge cx \cup dx' \neq 0 \Leftrightarrow (\exists p)(\exists q)(\exists t)\big(p \neq 0 \wedge q \neq 0 \wedge ab \leq p \leq a \cup b \\ \wedge\, p(a' \cup c)(b' \cup d) \cup p'(a \cup c)(b \cup d) \leq q \leq p(ac \cup bd) \cup p'(a'c \cup b'd) \\ \wedge\, x = ((a + p) \cup (c + q))' t \cup ((b + p) \cup (d + q)) t'\big). \end{gathered}$$

Example 1. *Solve the sistem*

$$mx \cup m'x' \neq 0 \wedge m'x \cup mx' \neq 0.$$

In acordance with Corrolary 1 we have $a = m, b = m', c = m', d = m$ *and*

$$mx \cup m'x' \neq 0 \wedge m'x \cup mx' \neq 0$$

$$\Leftrightarrow (\exists p)(\exists q)(\exists t)(p \neq 0 \wedge q \neq 0 \wedge 0 \leq p \leq 1 \wedge \ p' \leq q \leq p'$$

$$\wedge\ x = ((m + p) \cup (m' + q))'t \cup ((m' + p) \cup (m + q))t')$$

$$\Leftrightarrow (\exists p)(\exists t)(p \neq 0 \ \wedge \ p \neq 1 \wedge x = (m + p)'t \cup (m' + p)t').$$

Let $B = \{0, 1, m, m'\}$. *Conditions* $p \neq 0$ *and* $p \neq 1$ *imply* $p \in \{m, m'\}$.

If $p = m$ then $x = 1$. If $p = m'$ then $x = 0$. Therefore the set of all the solutions of the system in $\{0, 1, m, m'\}$ is $\{0,1\}$.

Example 2. *Let* $B = \{0, 1, m, m'\}$. *Solve the system*

$$mx \neq \ 0 \wedge \ mx' \neq 0. \tag{4}$$

Since $a = m, b = 0, c = 0, d = m$ *we have*

$$mx' \neq \ 0 \wedge \ mx \neq 0$$

$$\Leftrightarrow (\exists p)(\exists q)(\exists t)\big(p \neq 0 \wedge q \neq 0 \wedge 0 \leq p \leq m \wedge \ mp' \leq q \leq mp'$$

$$\wedge\ x = ((m + p) \cup q)'t \cup (p \cup (m + q))t'\big)$$

$$\Leftrightarrow (\exists p)(\exists q)(\exists t)\big(p \neq 0 \wedge q \neq 0 \wedge p \leq m \wedge q = mp' \ \wedge \ x = (m + p)'t \cup pt'\big).$$

Suppose that $B = \{0, 1, m, m'\}$. *Then* $p \leq m$ *and* $p \neq 0$ *imply* $p = m$. *Thus* $q = mp' = 0$. *Since* $q \neq 0 \wedge \ q = 0 \Leftrightarrow \perp$ *we get* $mx' \neq \ 0 \wedge \ mx \neq 0 \Leftrightarrow \perp$ *i.e. the system (4) has no solution in Boolean algebra* $\{0, 1, m, m'\}$.

Example 3. *Let us solve the system*

$$e'x \neq 0 \wedge ex' \neq 0.$$

Since $a = e', b = 0, c = 0, d = e$ *we have*

$$
\begin{aligned}
& e'x \neq 0 \wedge ex' \neq 0 \\
\Leftrightarrow\ & (\exists p)(\exists q)(\exists t)\big(p \neq 0 \wedge q \neq 0 \wedge 0 \leq p \leq e' \wedge pe \leq q \leq p'e \\
& \wedge x = ((e' + p) \cup q))'t \cup (p \cup (e + q))t'\big) \\
\Leftrightarrow\ & (\exists p)(\exists q)(\exists t)\big(0 \neq p \leq e' \wedge 0 \neq q \leq e \wedge x = (e'p' \cup q)'t \cup (eq' \cup p)t'\big) \\
\Leftrightarrow\ & (\exists p)(\exists q)\big(0 \neq p \leq e' \wedge 0 \neq q \leq e \wedge x = eq' \cup p\big).
\end{aligned}
$$

Suppose that $B = \{0, k, l, m, k', l', m', 1\}$ *and* $e = k'$. *Then*

$$k'x \neq 0 \wedge kx' \neq 0 \Leftrightarrow (\exists p)(\exists q)\big(0 \neq p \leq k \wedge 0 \neq q \leq k' \wedge x = k'q' \cup p\big).$$

The condition $0 \neq p \leq k$ *implies* $p = k$. *The condition* $0 \neq q \leq k'$ *implies* $q \in \{k', m, l\}$.

Taking $q = k'$ *we get* $x = k'k \cup k = k$.
Taking $q = m$ *we get* $x = k'm' \cup k = l \cup k = m'$.
Taking $q = l$ *we get* $x = k'l' \cup k = m \cup k = l'$.

The set of all the solutions of the system $e'x \neq 0 \wedge ex' \neq 0$ *in* $\{0, k, l, m, k', l', m', 1\}$ *is* $\{k, m', l'\}$.

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