mathematics

MDPI

Article

On Some Recent Results Concerning *F*-Suzuki-Contractions in b-Metric Spaces

Ersin Gilić [1,*], Diana Dolićanin-Đekić [1,2], Zoran D. Mitrović [3,4,*], Dženis Pučić [1] and Hassen Aydi [5,6]

1 Department of Mathematics, State University of Novi Pazar Vuka Karadžića bb, 36300 Novi Pazar, Serbia; diana.dolicanin@pr.ac.rs (D.D.-Đ.); dzpucic@np.ac.rs (D.P.)
2 Faculty of Technical Sciences, Kneza Miloša 7, 38220 Kosovska Mitrovica, Serbia
3 Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City 758307, Vietnam
4 Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City 758307, Vietnam
5 Institut Supérieur d'Informatique et des Techniques de Communication, Université de Sousse, H. Sousse 4000, Tunisia; hassen.aydi@isima.rnu.tn
6 China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
* Correspondence: egilic@np.ac.rs (E.G.); zoran.mitrovic@tdtu.edu.vn (Z.D.M.)

Received: 18 May 2020; Accepted: 3 June 2020; Published: 8 June 2020

check for updates

Abstract: The purpose of this manuscript is to provide much simpler and shorter proofs of some recent significant results in the context of generalized F-Suzuki-contraction mappings in b-complete b-metric spaces. By using our new approach for the proof that a Picard sequence is b-Cauchy, our results generalize, complement and improve many known results in the existing literature. Further, some new contractive conditions are provided here to illustrate the usability of the obtained theoretical results.

Keywords: Banach principle; F-Suzuki-contractive mapping; *b*-metric space; fixed point

1. Introduction and Preliminaries

It will be almost 100 years since S. Banach gave us one of the most beautiful achievements in the intellectual activity of modern man. This is his theorem proved in his doctoral dissertation in 1922. Recall this famous accomplishment: Each mapping T of the complete metric space (X, d) into itself, if it satisfies the condition that there exists $\lambda \in [0, 1)$ such that

$$d(T\omega, T\Omega) \leq \lambda d(\omega, \Omega), \tag{1}$$

for all $\omega, \Omega \in X$, then there is only one point $z \in X$ such that $T(z) = z$. More than that, for each $\Omega \in X$ a sequence of iterations $\Omega_{n+1} = T\Omega_n,\ n = 1, 2, 3, \ldots\ \Omega_1 = \Omega$, converges to such a fixed point z. The mapping that satisfies (1) is called a contraction.

Since that famous 1922, a number of mathematicians have been interested in this significant theorem, generalizing it in several directions. Some of them distorted the axioms of the metric space, while others distorted the condition (1). For all these generalizations, see References [1–23].

It is well known that the Banach contraction principle [24] is one of the most importance and attractive result in nonlinear analysis and in mathematical analysis, in general. Also, whole the fixed point theory is the significant subject in different fields as in geometry, differential equations, informatics, physics, economic, engineering, and so forth. After the existence of the solutions is ensured, the numerical methodology is applied to find the approximated solution. Fixed point of a function depends heavily on the considered spaces that are defined using the intuitive axioms. In particular, variant generalized metric spaces are given, like partial metric spaces, b-metric spaces, partial b-metric spaces, extended b-metric spaces, G-metric spaces, G_b-metric spaces, S-metric spaces, S_b-metric space, cone metric spaces, cone b-metric spaces, fuzzy metric space, fuzzy b-metric space, probabilistic metric space, and so forth. For more details of all generalized metric spaces, see References [25–28]. Different spaces will result in different types of fixed point results. On the other hand, there are a lot of different types of fixed point theorems in the literature.

The Banach contraction principle [24] is generalized by many authors in several directions (see References [25,26,28,29]). Some authors generalized it also in the context of cone metric spaces over Banach space, as well as, in the context of cone metric spaces over Banach algebra [30].

Fixed point theory is one of the major research areas in nonlinear analysis. This is partly due to the fact that in many real-world problems, fixed point technique is the basic mathematical tool used to ensure the existence of solutions which arise naturally in applications. As a consequence, fixed point theory is an essential area of study in applied and pure mathematics.The notion of a distance between two objects plays a important role, not only in mathematical sense, but also in its related fields.

The French mathematician Frechet initiated the study of metric spaces in Reference [31], Bakhtin [32] introduced b-metric spaces and gave the contraction mapping, which was the generalization of the Banach contraction principle. In 1993, Czerwik [33] extended this concept of b-metric spaces, whereas Shukla [34] introduced partial b-metrics in 2014. The concept of partial-metric spaces was introduced by Matthews [35] in 1994 as a generalization of standard metric spaces by replacing the condition $d(\Omega,\Omega) = 0$ with the condition $d(\Omega,\Omega) \leq d(\Omega,\omega)$, for all Ω,ω.

Throughout this manuscript, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ denote the set of real numbers, the set of all non-negative real numbers and the set of all positive integers, respectively.

Definition 1 ([32,33])**.** *A b-metric on a nonempty set X is a function $\theta : X \times X \to [0,+\infty)$ so that for all $\Omega,\omega,\tau \in X$ and a real $s \geq 1$, we have:*

(b1) $\theta(\Omega,\omega) = 0$ *if and only if* $\Omega = \omega$;
(b2) $\theta(\Omega,\omega) = \theta(\omega,\Omega)$;
(b3) $\theta(\Omega,\tau) \leq s(\theta(\Omega,\omega) + \theta(\omega,\tau))$.

The pair (X,θ) is then called a b-metric space with coefficient s.

The definitions of convergent and Cauchy sequence are formally the same in metric and b-metric spaces. In a b-metric space (X,θ), the following assertions are verified:

(a) A convergent sequence possesses a unique limit;
(b) Each convergent sequence is Cauchy;
(c) A b-metric is not necessarily continuous;
(d) A b-metric does not induce in general a topology on X;
(e) The b-metric space (X,θ) is b-complete if every b-Cauchy sequence in X is convergent in X;

Now, we will recall some definitions and lemmas which are essential to the proofs of fixed point theorems in the framework of b-metric spaces.

Lemma 1 ([36])**.** *Let (X,θ) be a b-metric space with $s \geq 1$, and $\{\Omega_n\}$ a sequence in X such that $\lim_{n\to+\infty} \theta(\Omega_n,\Omega_{n+1}) = 0$. If $\{\Omega_n\}$ is not a b-Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences of positive integers $\{n(k)\}$ and $\{m(k)\}$ such that the following*

$$\left\{\theta\left(\Omega_{m(k)},\Omega_{n(k)}\right)\right\}, \left\{\theta\left(\Omega_{m(k)},\Omega_{n(k)+1}\right)\right\}, \left\{\theta\left(\Omega_{m(k)+1},\Omega_{n(k)}\right)\right\}, \left\{\theta\left(\Omega_{m(k)+1},\Omega_{n(k)+1}\right)\right\}, \quad (2)$$

exist and verify

$$\varepsilon \leq \lim \inf_{k\to+\infty} \theta\left(\Omega_{m(k)},\Omega_{n(k)}\right) \leq \lim \sup_{k\to+\infty} \theta\left(\Omega_{m(k)},\Omega_{n(k)}\right) \leq \varepsilon s,$$

$$\frac{\varepsilon}{s} \leq \lim_{k\to+\infty}\inf \theta\left(\Omega_{m(k)}, \Omega_{n(k)+1}\right) \leq \lim_{k\to+\infty}\sup \theta\left(\Omega_{m(k)}, \Omega_{n(k)+1}\right) \leq \varepsilon s^2,$$

$$\frac{\varepsilon}{s} \leq \lim_{k\to+\infty}\inf \theta\left(\Omega_{m(k)+1}, \Omega_{n(k)}\right) \leq \lim_{k\to+\infty}\sup \theta\left(\Omega_{m(k)+1}, \Omega_{n(k)}\right) \leq \varepsilon s^2,$$

$$\frac{\varepsilon}{s^2} \leq \lim_{k\to+\infty}\inf \theta\left(\Omega_{m(k)+1}, \Omega_{n(k)+1}\right) \leq \lim_{k\to+\infty}\sup \theta\left(\Omega_{m(k)+1}, \Omega_{n(k)+1}\right) \leq \varepsilon s^3.$$

Corollary 1. *Putting $s = 1$ in Lemma 1 we obtain that all the sequences from (2) tend to ε^+ when $k \to +\infty$ (see also Lemma 2.1. of Reference [36]).*

Lemma 2 ([36])**.** *Let (X, θ) be a b-metric space with $s \geq 1$, and assume that $\{\Omega_n\}$ and $\{\omega_n\}$ are b-convergent to the limits Ω and ω, respectively. Then we have*

$$\frac{1}{s^2}\theta(\Omega, \omega) \leq \lim_{n\to+\infty}\inf \theta(\Omega_n, \omega_n) \leq \lim_{n\to+\infty}\sup \theta(\Omega_n, \omega_n) \leq s^2\theta(\Omega, \omega).$$

In particular, if $\Omega = \omega$, then we have $\lim_{n\to+\infty} \theta(\Omega_n, \omega_n) = 0$. Moreover, for each $\sigma \in X$, we have

$$\frac{1}{s}\theta(\Omega, \sigma) \leq \lim_{n\to+\infty}\inf \theta(\Omega_n, \sigma) \leq \lim_{n\to+\infty}\sup \theta(\Omega_n, \sigma) \leq s\theta(\Omega, \sigma).$$

Instead of using the previous lemmas, in this manuscript, we will show that much more subtle and convenient is the next recent result.

Lemma 3 ([37–39])**.** *Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a sequence in a b-metric space (X, θ) with $s \geq 1$, such that*

$$\theta(\Omega_n, \Omega_{n+1}) \leq \mu\theta(\Omega_{n-1}, \Omega_n), \tag{3}$$

for some $\mu \in [0, 1)$, and each $n \in \mathbb{N}$. Then $\{\Omega_n\}$ is a b-Cauchy sequence in (X, θ).

In Reference [40], Wardowski defined a new type of mappings as follows:

Definition 2. *Let $\mathcal{F}$ be the family of all functions $F : (0, +\infty) \to (-\infty, +\infty)$ so that:*

(F1) *for all $\nu, \eta \in (0, +\infty)$ such that $\nu < \eta$, $F(\nu) < F(\eta)$, that is, (F is strictly increasing);*
(F2) *for each sequence $\{\alpha_n\}_{n=1}^{+\infty}$ of positive numbers, $\lim_{n\to+\infty} \upsilon_n = 0$ if and only if $\lim_{n\to+\infty} F(\upsilon_n) = 0$;*
(F3) *there exists $\kappa \in (0, 1)$ such that $\lim_{\upsilon\to 0^+} \alpha^\kappa F(\upsilon) = 0$.*

Definition 3 ([40])**.** *Let (X, θ) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction on (X, θ) if there exist $F \in \mathcal{F}$ and $\tau > 0$ so that, for all $\Omega, \omega \in X$,*

$$\theta(T\Omega, T\omega) > 0 \text{ implies } \tau + F(\theta(T\Omega, T\omega)) \leq F(\theta(\Omega, \omega)). \tag{4}$$

Wardowski [40] gave a new generalization of Banach contraction principle as follows:

Theorem 1 ([40])**.** *Let (X, θ) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point $\Omega^* \in X$ and for every $\Omega \in X$ the sequence $\{T^n\Omega\}_{n=1}^{+\infty}$ converges to Ω^*.*

In 2014, Wardowski and Dung [41] initiated the concept of an F-weak contraction and established the following related fixed point result

Definition 4 ([41])**.** *A mapping $T : X \to X$ is said to be an F-weak contraction on the metric space (X, θ) if there are $F \in \mathcal{F}$ and $\tau > 0$ so that, for all $\Omega, \omega \in X$,*

$$\theta (T\Omega, T\omega) > 0 \text{ implies } \tau + F (\theta (T\Omega, T\omega)) \leq F (M (\Omega, \omega)), \tag{5}$$

where

$$M (\Omega, \omega) = \max \left\{ \theta (\Omega, \omega), \theta (\Omega, T\Omega), \theta (\omega, T\omega), \frac{\theta (\Omega, T\omega) + \theta (\omega, T\Omega)}{2} \right\}.$$

Theorem 2 ([41])**.** *Let (X, θ) be a complete metric space and let $T : X \to X$ be an F-weak contraction. If T or F is continuous, then T possesses a unique fixed point $\Omega^* \in X$ and for every $\Omega \in X$ the sequence $\{T^n \Omega\}_{n=1}^{+\infty}$ is convergent to Ω^*.*

Recall that a contraction condition for a self-mapping T on a metric space (X, θ), usually contained at most five values $\theta (\Omega, \omega), \theta (\Omega, T\Omega), \theta (\omega, T\omega), \theta (\Omega, T\omega), \theta (\omega, T\Omega)$ (for example see References [42,43]). Recently, by adding the following four new values $\theta \left(T^2\Omega, \Omega\right)$, $\theta \left(T^2\Omega, T\Omega\right)$, $\theta \left(T^2\Omega, \omega\right)$, $\theta \left(T^2\Omega, T\omega\right)$ to a contraction condition, Dung and Hang ([44]) proved some fixed point theorems. They gave examples to show that their result is a real generalization of known ones in exiting literature.

Definition 5 ([44])**.** *A mapping $T : X \to X$ is said to be a generalized F-contraction on the metric space (X, θ) if there are $F \in \mathcal{F}$ and $\tau > 0$ so that, for all $\Omega, \omega \in X$,*

$$\theta (T\Omega, T\omega) > 0 \text{ implies } \tau + F (\theta (T\Omega, T\omega)) \leq F (N (\Omega, \omega)), \tag{6}$$

where

$$N (\Omega, \omega) = \max \left\{ \theta (\Omega, \omega), \theta (\Omega, T\Omega), \theta (\omega, T\omega), \frac{\theta (\Omega, T\omega) + \theta (\omega, T\Omega)}{2}, \right.$$
$$\left. \frac{\theta \left(T^2\Omega, \Omega\right) + \theta \left(T^2\Omega, T\omega\right)}{2}, \theta \left(T^2\Omega, T\Omega\right), \theta \left(T^2\Omega, \omega\right), \theta \left(T^2\Omega, T\omega\right) \right\}.$$

Theorem 3 ([44])**.** *Let $T : X \to X$ be a generalized F-contraction mapping on a complete metric space (X, θ). If T or F is continuous, then T possesses a unique fixed point $\Omega^* \in X$ and for each $\Omega \in X$, the sequence $\{T^n \Omega\}_{n=1}^{+\infty}$ is convergent to Ω^*.*

In 2014, Piri and Kumam [45] described a large set of functions by replacing the condition (F3) in the definition of an F-contraction introduced by Wardowski [40] by the following:

(F3') F is continuous on $(0, +\infty)$.

They denote by F the set of all functions $F : (0, +\infty) \to (-\infty, +\infty)$ satisfying the conditions **(F1)**, **(F2)** and **(F3')**. Denote by Ψ the set of all functions $\psi : [0, +\infty) \to [0, +\infty)$ so that ψ is continuous and $\psi (t) = 0$ if and only $t = 0$. Under this new set-up, Piri and Kumam introduced and established the following Wardowski and Suzuki type fixed point results in b-metric spaces.

Definition 6 ([45])**.** *A self mapping T on a b-metric space (X, θ) is said to be a generalized F-Suzuki-contraction if there is $F : (0, +\infty) \to (-\infty, +\infty)$ satisfying **(F1)** and **(F3')** such that, for all $\Omega, \omega \in X$ with $\Omega \neq \omega$,*

$$\frac{1}{2s} \theta (\Omega, T\Omega) < \theta (\Omega, \omega) \text{ implies } F \left(s^5 \theta (T\Omega, T\omega)\right) \leq F (M_T (\Omega, \omega)) - \psi (M_T (\Omega, \omega)), \tag{7}$$

where $\psi \in \Psi$ and

$$M_T (\Omega, \omega) = \max \left\{ \theta (\Omega, \omega), \theta \left(T^2\Omega, \omega\right), \frac{\theta (\Omega, T\omega) + \theta (\omega, T\Omega)}{2s}, \right.$$

$$\frac{\theta\left(T^2\Omega,\Omega\right)+\theta\left(T^2\Omega,T\omega\right)}{2s},\theta\left(T^2\Omega,T\omega\right)+\theta\left(T^2\Omega,T\Omega\right),$$
$$\theta\left(T^2\Omega,T\omega\right)+\theta\left(T\Omega,\Omega\right),\ \theta\left(T\Omega,\omega\right)+\theta\left(\omega,T\omega\right)\Big\}.$$

Theorem 4 ([45])**.** *Let $T : X \to X$ be a generalized F-Suzuki-contraction on a complete b-metric space (X, θ). Then T possesses a unique fixed point $\Omega^* \in X$ and for each $\Omega \in X$ the sequence $\{T^n\Omega\}_{n=1}^{+\infty}$ is convergent to Ω^*.*

2. Main Results

In the sequel of this manuscript, the function $F : (0, +\infty) \to (-\infty, +\infty)$ only verifies the condition (F1), while $\psi \in \Psi$. Our first result generalizes and improves Theorem 3. Namely, first of all we introduce the following:

Definition 7. *A mapping $T : X \to X$ is said to be a generalized F1-contraction on a metric space (X, θ) if there are $F : (0, +\infty) \to (-\infty, +\infty)$ satisfying the condition (F1) and $\tau > 0$ so that, for all $\Omega, \omega \in X$,*

$$\theta\left(T\Omega,T\omega\right)>0 \text{ implies } \tau+F\left(\theta\left(T\Omega,T\omega\right)\right)\leq F\left(N\left(\Omega,\omega\right)\right), \tag{8}$$

in which

$$N\left(\Omega,\omega\right)=\max\Bigg\{\theta\left(\Omega,\omega\right),\ \theta\left(\Omega,T\Omega\right),\ \theta\left(\omega,T\omega\right),\frac{\theta\left(\Omega,T\omega\right)+\theta\left(\omega,T\Omega\right)}{2},$$
$$\frac{\theta\left(T^2\Omega,\Omega\right)+\theta\left(T^2\Omega,T\omega\right)}{2},\ \theta\left(T^2\Omega,T\Omega\right),\ \theta\left(T^2\Omega,\omega\right),\ \theta\left(T^2\Omega,T\omega\right)\Bigg\}.$$

Theorem 5. *Let (X, θ) be a complete metric space and $T : X \to X$ be a generalized F1-contraction on (X, θ). Then T possesses a unique fixed point $\Omega^* \in X$ and for each $\Omega \in X$ the sequence $\{T^n\Omega\}_{n=1}^{+\infty}$ converges to Ω^*.*

Proof. First, we will to check that the condition (8) gives the uniqueness of the fixed point if it exists. Indeed, let $\overline{\Omega}$ and $\overline{\omega}$ be two distinct fixed points of T. This means that the following holds true:

$$\tau+F\left(\theta\left(T\overline{\Omega},T\overline{\omega}\right)\right)\leq F\left(N\left(\overline{\Omega},\overline{\omega}\right)\right),$$

where

$$N\left(\overline{\Omega},\overline{\omega}\right)=\max\Bigg\{\theta\left(\overline{\Omega},\overline{\omega}\right),\ \theta\left(\overline{\Omega},T\overline{\Omega}\right),\ \theta\left(\overline{\omega},T\overline{\omega}\right),\frac{\theta\left(\overline{\Omega},T\overline{\omega}\right)+\theta\left(\overline{\omega},T\overline{\Omega}\right)}{2},$$
$$\frac{\theta\left(T^2\overline{\Omega},\overline{\Omega}\right)+\theta\left(T^2\overline{\Omega},T\overline{\omega}\right)}{2},\ \theta\left(T^2\overline{x},T\overline{\Omega}\right),\ \theta\left(T^2\overline{\Omega},\overline{\omega}\right),\ \theta\left(T^2\overline{\Omega},T\overline{\omega}\right)\Bigg\},$$

that is,

$$N\left(\overline{\Omega},\overline{\omega}\right)=\max\Bigg\{\theta\left(\overline{\Omega},\overline{\omega}\right),0,0,\frac{\theta\left(\overline{\Omega},\overline{\omega}\right)+\theta\left(\overline{\omega},\overline{\Omega}\right)}{2},\frac{0+\theta\left(\overline{\Omega},\overline{\omega}\right)}{2},$$
$$0,\ \theta\left(\overline{\Omega},\overline{\omega}\right),\ \theta\left(\overline{\Omega},\overline{\omega}\right)\Big\}=\theta\left(\overline{\Omega},\overline{\omega}\right).$$

Further, we get $\tau+F\left(\theta\left(\overline{\Omega},\overline{\omega}\right)\right)\leq F\left(\theta\left(\overline{\Omega},\overline{\omega}\right)\right)$. It is a contradiction. Hence, the proof of the uniqueness of the fixed point for mapping T, if it exists, is completed.

In order to show that T has a fixed point, let Ω_0 be arbitrary point in X. Now, we define a Picard's sequence $\{\Omega_n\}_{n\in\mathbb{N}\cup\{0\}}, \Omega_{n+1} = T\Omega_n$. If $\Omega_p = \Omega_{p+1}$ for some $p \in \mathbb{N}\cup\{0\}$ then Ω_p is a unique fixed point and the proof is finished. Therefore, suppose now that $\Omega_n \neq \Omega_{n+1}$ for all $n \in \mathbb{N}\cup\{0\}$. According to the condition (8), it follows that

$$\tau+F\left(\theta\left(T\Omega_{n-1},T\Omega_n\right)\right)\leq F\left(N\left(\Omega_{n-1},\Omega_n\right)\right),$$

where

$$
\begin{aligned}
& N\left(\Omega_{n-1},\Omega_{n}\right) \\
= \; & \max\left\{\theta\left(\Omega_{n-1},\Omega_{n}\right),\,\theta\left(\Omega_{n-1},\Omega_{n}\right),\,\theta\left(\Omega_{n},\Omega_{n+1}\right),\,\frac{\theta\left(\Omega_{n-1},\Omega_{n+1}\right)+\theta\left(\Omega_{n},\Omega_{n}\right)}{2},\right. \\
& \left.\frac{\theta\left(\Omega_{n+1},\Omega_{n-1}\right)+\theta\left(\Omega_{n+1},\Omega_{n+1}\right)}{2},\,\theta\left(\Omega_{n+1},\Omega_{n}\right),\,\theta\left(\Omega_{n+1},\Omega_{n}\right),\,\theta\left(\Omega_{n+1},\Omega_{n+1}\right)\right\},
\end{aligned}
$$

that is,

$$N\left(\Omega_{n-1},\Omega_{n}\right)\leq\max\left\{\theta\left(\Omega_{n-1},\Omega_{n}\right),\theta\left(\Omega_{n},\Omega_{n+1}\right)\right\}.$$

Hence, we obtain by (8) that

$$\tau+F\left(\theta\left(\Omega_{n},\Omega_{n+1}\right)\right)\leq F\left(\max\left\{\theta\left(\Omega_{n-1},\Omega_{n}\right),\theta\left(\Omega_{n},\Omega_{n+1}\right)\right\}\right). \tag{9}$$

If $\max\left\{\theta\left(\Omega_{n-1},\Omega_{n}\right),\theta\left(\Omega_{n},\Omega_{n+1}\right)\right\}=\theta\left(\Omega_{n},\Omega_{n+1}\right)$ then from (9) it follows a contradiction. Therefore,

$$\tau+F\left(\theta\left(\Omega_{n},\Omega_{n+1}\right)\right)\leq F\left(\theta\left(\Omega_{n-1},\Omega_{n}\right)\right), \tag{10}$$

for all $n\in\mathbb{N}$. Further, according to the (10) and the condition (F1) we obtain that $\theta\left(\Omega_{n},\Omega_{n+1}\right)<\theta\left(\Omega_{n-1},\Omega_{n}\right)$ for all $n\in\mathbb{N}$. This further means that there exists $\rho^{*}=\lim_{n\to+\infty}\theta\left(\Omega_{n},\Omega_{n+1}\right)$. In this case, (10) implies

$$\tau+F\left(\rho^{*}+0\right)\leq F\left(\rho^{*}+0\right),$$

which is a contradiction with $\tau>0$.

Now, we prove that $\{\Omega_n\}_{n\in\mathbb{N}\cup\{0\}}$, is a Cauchy sequence by supposing contrary. Putting $\Omega=\Omega_{n(k)},\omega=\Omega_{m(k)}$ in (6), we get

$$\tau+F\left(\theta\left(\Omega_{n(k)+1},\Omega_{m(k)+1}\right)\right)\leq F\left(N\left(\Omega_{n(k)},\Omega_{m(k)}\right)\right), \tag{11}$$

where

$$
\begin{aligned}
N\left(\Omega_{n(k)},\Omega_{m(k)}\right) = \max\Big\{ & \theta\left(\Omega_{n(k)},\Omega_{m(k)}\right),\,\theta\left(\Omega_{n(k)},\Omega_{n(k)+1}\right),\,\theta\left(\Omega_{m(k)},\Omega_{m(k)+1}\right), \\
& \frac{\theta\left(\Omega_{n(k)},\Omega_{m(k)+1}\right)+\theta\left(\Omega_{m(k)},\Omega_{n(k)+1}\right)}{2},\,\frac{\theta\left(\Omega_{n(k)+2},\Omega_{n(k)}\right)+\theta\left(\Omega_{n(k)+2},\Omega_{m(k)}\right)}{2}, \\
& \theta\left(\Omega_{n(k)+2},\Omega_{n(k)+1}\right),\,\theta\left(\Omega_{n(k)+2},\Omega_{m(k)}\right),\,\theta\left(\Omega_{n(k)+2},\Omega_{m(k)+1}\right)\Big\}.
\end{aligned}
$$

Since, by Corollary 1. $\theta\left(\Omega_{n(k)+1},\Omega_{m(k)+1}\right),\theta\left(\Omega_{n(k)},\Omega_{m(k)}\right)$ and $N\left(\Omega_{n(k)},\Omega_{m(k)}\right)$ tend to ε^{+} as $k\to+\infty$, we obtain from (11) that

$$\tau+F\left(\varepsilon^{+}+0\right)\leq F\left(\varepsilon^{+}+0\right),$$

which is a contradiction. Hence the sequence $\{\Omega_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy.

Since (X,θ) is a complete metric space, we have that the sequence $\{\Omega_n\}_{n\in\mathbb{N}\cup\{0\}}$ converges to some $\Omega^{*}\in X$.

If the mapping T is continuous then $T\Omega_n$ tends to $T\Omega^{*}$, that is, Ω^{*} is a unique fixed point of T.

In the case that F is continuous, we have the following:

Firstly, we can suppose that both $\Omega^{*},T\Omega^{*}\notin\{\Omega_n\}_{n\in\mathbb{N}\cup\{0\}}$. This is a consequence of the fact that $\theta\left(\Omega_{n},\Omega_{n+1}\right)<\theta\left(\Omega_{n-1},\Omega_{n}\right)$ for all $n\in\mathbb{N}\cup\{0\}$, which implies that $\Omega_n\neq\Omega_m$ whenever $n\neq m$.

Putting $\Omega = \Omega_n$ and $\omega = \Omega^*$ in (8) and supposing that $\theta(\Omega^*, T\Omega^*) > 0$, we immediately obtain

$$\tau + F(\theta(\Omega_{n+1}, T\Omega^*)) \leq F(N(\Omega_n, \Omega^*)), \tag{12}$$

in which

$$N(\Omega_n, \Omega^*) = \max\left\{\theta(\Omega_n, \Omega^*), \theta(\Omega_n, \Omega_{n+1}), \theta(\Omega^*, T\Omega^*), \frac{\theta(\Omega_n, T\Omega^*) + \theta(\Omega^*, \Omega_{n+1})}{2},\right.$$

$$\left.\frac{\theta(\Omega_{n+2}, \Omega_n) + \theta(\Omega_{n+2}, T\Omega^*)}{2}, \theta\ (\Omega_{n+2}\Omega_{n+1}), \theta(\Omega_{n+2}, \Omega^*), \theta(\Omega_{n+2}, T\Omega^*)\right\} \to \theta(\Omega^*, T\Omega^*)\}$$

when $n \to +\infty$. Since the function F is continuous, (12) give us

$$\tau + F(\theta(\Omega^*, T\Omega^*)) \leq F(\theta(\Omega^*, T\Omega^*)),$$

which is a contradiction with $\tau > 0$. Hence, the assumption that $\theta(\Omega^*, T\Omega^*) > 0$ is wrong. So, the point Ω^* is the unique fixed point of T. □

Immediately consequence of Theorem 5 are the following new contractive conditions which complement ones from References [42,43].

Corollary 2. *Let (X, θ) be a complete metric space and let $T : X \to X$ be a generalized $F1-$contraction so that there is $\tau > 0$ and for all $\Omega, \omega \in X$ with $\theta(T\Omega, T\omega) > 0$ the following implications hold true:*

$$\tau + \theta(T\Omega, T\omega) \leq N(\Omega, \omega)$$

$$\tau + \exp(\theta(T\Omega, T\omega)) \leq \exp(N(\Omega, \omega))$$

$$\tau - \frac{1}{\theta(T\Omega, T\omega)} \leq -\frac{1}{N(\Omega, \omega)}$$

$$\tau - \frac{1}{\theta(T\Omega, T\omega)} + \theta(T\Omega, T\omega) \leq -\frac{1}{N(\Omega, \omega)} + N(\Omega, \omega)$$

$$\tau + \frac{1}{1 - \exp(\theta(T\Omega, T\omega))} \leq \frac{1}{1 - \exp(N(\Omega, \omega))}$$

$$\tau + \frac{1}{\exp(-\theta(T\Omega, T\omega)) - \exp(\theta(T\Omega, T\omega))} \leq \frac{1}{\exp(-N(\Omega, \omega)) - \exp(N(\Omega, \omega))},$$

in which $N(\Omega, \omega)$ is given as in Definition 7. Then in each of these cases T has a unique fixed point Ω^* and for all $x \in X$ the sequence $\{T^n x\}_{n=1}^{+\infty}$ converges to Ω^*.

Proof. Since, each of the following functions

$$r \mapsto r,\ r \mapsto e^r,\ r \mapsto -\frac{1}{r},\ r \mapsto -\frac{1}{r} + r,\ r \mapsto \frac{1}{1 - e^r},\ r \mapsto \frac{1}{e^{-r} - e^r}$$

is strictly increasing on $(0, +\infty)$ then the proof follows by Theorem 5. □

Now, we prove our second new result as the improvement and proper generalization of Theorem 4 (Theorem 2.2, [45]).

Theorem 6. *Let $T : X \to X$ be a generalized $F-$Suzuki-contraction on a complete b-metric space $(X, d, s > 1)$. Then T possesses a unique fixed point $\Omega^* \in X$ and for each $\Omega \in X$ the sequence $\{T^n\Omega\}_{n=1}^{+\infty}$ is convergent to Ω^*.*

Proof. The condition (7) becomes

$$\Omega \neq \omega \text{ and } \frac{1}{2s}\theta(\Omega, T\Omega) < \theta(\Omega, \omega) \text{ implies } \theta(T\Omega, T\omega) \leq \frac{1}{s^m} M_T(\Omega, \omega), \tag{13}$$

in which $m \in (0, 5]$ and

$$M_T(\Omega, \omega) = \max\left\{\theta(\Omega, \omega), \theta\left(T^2\Omega, \omega\right), \frac{\theta(\Omega, T\omega) + \theta(\omega, T\Omega)}{2s}, \frac{\theta(T^2\Omega, \Omega) + \theta(T^2\Omega, T\omega)}{2s}, \theta\left(T^2\Omega, T\omega\right) + \theta\left(T^2\Omega, T\Omega\right), \theta\left(T^2\Omega, T\omega\right) + \theta(T\Omega, \Omega), \theta(T\Omega, \omega) + \theta(\omega, T\omega)\right\}.$$

The proof further follows in several steps.

Step 1. The uniqueness.

If Ω^*, ω^* are two distinct fixed points of T, then $\Omega^* \neq \omega^*$ and $\frac{1}{2s}\theta(\Omega^*, T\Omega^*) = 0 < \theta(\Omega^*, \omega^*)$ holds true. Therefore, from (13) follows

$$\theta(\Omega^*, \omega^*) = \theta(T\Omega^*, T\omega^*) \leq \frac{1}{s^m} M_T(\Omega^*, \omega^*), \tag{14}$$

in which

$$M_T(\Omega^*, \omega^*) = \max\left\{\theta(\Omega^*, \omega^*), \theta\left(T^2\Omega^*, \omega^*\right), \frac{\theta(\Omega^*, T\omega^*) + \theta(\omega^*, T\Omega^*)}{2s}, \frac{\theta(T^2\Omega^*, \Omega^*) + \theta(T^2\Omega^*, T\omega^*)}{2s}, \theta\left(T^2\Omega^*, T\omega^*\right) + \theta\left(T^2\Omega^*, T\Omega^*\right), \theta\left(T^2\Omega^*, T\omega^*\right) + \theta(T\Omega^*, \Omega^*), \theta(T\Omega^*, \omega^*) + \theta(\omega^*, T\omega^*)\right\},$$

that is,

$$M_T(\Omega^*, \omega^*) = \max\left\{\theta(\Omega^*, \omega^*), \theta(\Omega^*, \omega^*), \frac{\theta(\Omega^*, \omega^*)}{s}, \frac{0 + \theta(\Omega^*, \omega^*)}{2s}, \theta(\Omega^*, \omega^*) + 0, \theta(\Omega^*, \omega^*) + 0, \theta(\Omega^*, \omega^*) + 0\right\} = \left\{\theta(\Omega^*, \omega^*), \frac{\theta(\Omega^*, \omega^*)}{s}, \frac{\theta(\Omega^*, \omega^*)}{2s}\right\} = \theta(\Omega^*, \omega^*).$$

Now, the condition (14) becomes $\theta(\Omega^*, \omega^*) \leq \frac{1}{s^m}\theta(\Omega^*, \omega^*)$, which is a contradiction. Hence, if T has a fixed point, it is unique.

Step 2. The sequence $\Omega_n = T^n\Omega_0$, $\Omega_0 \in X$ is b-Cauchy.

If $\Omega_p = \Omega_{p+1}$ for some $p \in \mathbb{N}$, the proof is finished. In this case, Ω_p is the unique fixed point. In this case, the sequence $\{\Omega_n\}_{n \in \mathbb{N} \cup \{0\}}$ is a b-Cauchy. Therefore, let $\Omega_n \neq \Omega_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Hence, we have that $\Omega_n \neq \Omega_{n+1}$ and $\frac{1}{2s}\theta(\Omega_n, T\Omega_n) < \theta(\Omega_n, \Omega_{n+1})$ ($\Omega = \Omega_n$ and $\omega = \Omega_{n+1}$), and therefore (13) becomes

$$\theta(\Omega_{n+1}, \Omega_{n+2}) = \theta(T\Omega_n, T\Omega_{n+1}) \leq \frac{1}{s^m} M_T(\Omega_n, \Omega_{n+1}), \tag{15}$$

where

$$M_T(\Omega_n, \Omega_{n+1}) = \max\left\{\theta(\Omega_n, \Omega_{n+1}), \theta(\Omega_{n+2}, \Omega_{n+1}), \frac{\theta(\Omega_n, \Omega_{n+2}) + \theta(\Omega_{n+1}, \Omega_{n+1})}{2s}, \right.$$

$$\frac{\theta\left(\Omega_{n+2},\Omega_{n}\right)+\theta\left(\Omega_{n+2},\Omega_{n+2}\right)}{2s},\theta\left(\Omega_{n+2},\Omega_{n+2}\right)+\theta\left(\Omega_{n+2},\Omega_{n+1}\right),$$
$$\theta\left(\Omega_{n+2},\Omega_{n+2}\right)+\theta\left(\Omega_{n+1},\Omega_{n}\right),\ \theta\left(\Omega_{n+1},\Omega_{n+1}\right)+\theta\left(\Omega_{n+1},\Omega_{n+2}\right)\},$$

that is,

$$M_T\left(\Omega_n,\Omega_{n+1}\right)\leq\max\left\{\theta\left(\Omega_n,\Omega_{n+1}\right),\theta\left(\Omega_{n+2},\Omega_{n+1}\right),\frac{\theta\left(\Omega_n,\Omega_{n+2}\right)}{2s},\theta\left(\Omega_{n+2},\Omega_{n+1}\right)\right\}$$
$$\leq\max\left\{\theta\left(\Omega_n,\Omega_{n+1}\right),\theta\left(\Omega_{n+2},\Omega_{n+1}\right)\right\}.$$

Now, from (15) we have

$$\theta\left(\Omega_{n+1},\Omega_{n+2}\right)\leq\frac{1}{s^m}\max\left\{\theta\left(\Omega_n,\Omega_{n+1}\right),\theta\left(\Omega_{n+2},\Omega_{n+1}\right)\right\}. \tag{16}$$

If $\max\left\{\theta\left(\Omega_n,\Omega_{n+1}\right),\theta\left(\Omega_{n+2},\Omega_{n+1}\right)\right\}=\theta\left(\Omega_{n+2},\Omega_{n+1}\right)$ we obtain a contradiction $1\leq\frac{1}{s^m}$. Because, (16) become

$$\theta\left(\Omega_{n+1},\Omega_{n+2}\right)\leq\mu\theta\left(\Omega_n,\Omega_{n+1}\right),\mu=\frac{1}{s^m}\in(0,1),$$

then according to Lemma 3, we have that the sequence $\Omega_n=T^n\Omega_0$ is a b-Cauchy.

Step 3. The existence of a fixed point.

Since b-metric space $(X,d,s>1)$ is b-complete, there exists a unique point $\Omega^*\in X$ such that the sequence $\Omega_n=T^n\Omega_0$ converges to Ω^*, that is, $\lim_{n\to+\infty}\theta\left(\Omega_n,\Omega^*\right)=0$.

Now, as in (Reference [45], page 8), we get the following two relations:

$$\frac{1}{2s}\theta\left(\Omega_n,T\Omega_n\right)<\theta\left(\Omega_n,\Omega^*\right)\ \text{or}\ \frac{1}{2s}\theta\left(T\Omega_n,T^2\Omega_n\right)<\theta\left(T\Omega_n,\Omega^*\right),$$

for all $n\in\mathbb{N}$. Also, since $\theta\left(\Omega_{n+1},\Omega_n\right)<\theta\left(\Omega_n,\Omega_{n-1}\right)$ for all $n\in\mathbb{N}$ we can suppose that both $\Omega^*,T\Omega^*\notin\left\{\Omega_n\right\}_{n\in\mathbb{N}\cup\{0\}}$. Therefore, if $\frac{1}{2s}\theta\left(\Omega_n,T\Omega_n\right)<\theta\left(\Omega_n,\Omega^*\right)$, the condition (13) becomes

$$\theta\left(T\Omega_n,T\Omega^*\right)\leq\frac{1}{s^m}M_T\left(\Omega_n,\Omega^*\right), \tag{17}$$

where

$$M_T\left(\Omega_n,\Omega^*\right)=\max\left\{\theta\left(\Omega_n,\Omega^*\right),\theta\left(\Omega_{n+2},\Omega^*\right),\frac{\theta\left(\Omega_{n+1},\Omega^*\right)+\theta\left(\Omega_n,T\Omega^*\right)}{2s},\right.$$
$$\frac{\theta\left(\Omega_{n+2},\Omega_n\right)+\theta\left(\Omega_{n+2},T\Omega^*\right)}{2s},\theta\left(\Omega_{n+2},T\Omega^*\right)+\theta\left(\Omega_{n+2},\Omega_{n+1}\right),$$
$$\theta\left(\Omega_{n+2},T\Omega^*\right)+\theta\left(\Omega_{n+1},\Omega_n\right),\ \theta\left(\Omega_{n+1},\Omega^*\right)+\theta\left(\Omega^*,T\Omega^*\right)\}.$$

Since $\theta\left(\Omega_k,T\Omega^*\right)\leq s\theta\left(\Omega_k,\Omega^*\right)+s\theta\left(\Omega^*,T\Omega^*\right)$ for $k=n$ and $n+2$, respectively, we obtain that

$$M_T\left(\Omega_n,\Omega^*\right)\leq\max\left\{\theta\left(\Omega_n,\Omega^*\right),\theta\left(\Omega_{n+2},\Omega^*\right),\frac{\theta\left(\Omega_{n+1},\Omega^*\right)+s\theta\left(\Omega_n,\Omega^*\right)+s\theta\left(\Omega^*,T\Omega^*\right)}{2s},\right.$$
$$\frac{\theta\left(\Omega_{n+2},\Omega_n\right)+s\theta\left(\Omega_{n+2},\Omega^*\right)+s\theta\left(\Omega^*,T\Omega^*\right)}{2s},s\theta\left(\Omega_{n+2},\Omega^*\right)+s\theta\left(\Omega^*,T\Omega^*\right)+\theta\left(\Omega_{n+2},\Omega_{n+1}\right),$$
$$s\theta\left(\Omega_{n+2},\Omega^*\right)+s\theta\left(\Omega^*,T\Omega^*\right)+\theta\left(\Omega_{n+1},\Omega_n\right),\ \theta\left(\Omega_{n+1},\Omega^*\right)+\theta\left(\Omega^*,T\Omega^*\right)\}$$
$$\to\max\left\{0,\frac{\theta\left(\Omega^*,T\Omega^*\right)}{2},s\theta\left(\Omega^*,T\Omega^*\right),\theta\left(\Omega^*,T\Omega^*\right)\right\}=s\theta\left(\Omega^*,T\Omega^*\right)\ \text{when}\ n\to+\infty. \tag{18}$$

On the other hand, if $\frac{1}{2s}\theta\left(T\Omega_n, T^2\Omega_n\right) < \theta\left(T\Omega_n, \Omega^*\right)$, that is, $\frac{1}{2s}\theta\left(\Omega_{n+1}, T\Omega_{n+1}\right) < \theta\left(\Omega_{n+1}, \Omega^*\right)$ from (13), it follows

$$\theta\left(T\Omega_{n+1}, T\Omega^*\right) \leq \frac{1}{s^m} M_T\left(\Omega_{n+1}, \Omega^*\right), \tag{19}$$

where

$$M_T\left(\Omega_{n+1}, \Omega^*\right) = \max\left\{\theta\left(\Omega_{n+1}, \Omega^*\right), \theta\left(\Omega_{n+3}, \Omega^*\right), \frac{\theta\left(\Omega_{n+1}, T\Omega^*\right) + \theta\left(\Omega^*, T\Omega^*\right)}{2s}, \right.$$
$$\frac{\theta\left(\Omega_{n+3}, \Omega_{n+1}\right) + \theta\left(\Omega_{n+3}, T\Omega^*\right)}{2s}, \theta\left(\Omega_{n+3}, T\Omega^*\right) + \theta\left(\Omega_{n+3}, \Omega_{n+2}\right),$$
$$\left.\theta\left(\Omega_{n+3}, T\Omega^*\right) + \theta\left(\Omega_{n+2}, \Omega_{n+1}\right), \theta\left(\Omega_{n+2}, \Omega^*\right) + \theta\left(\Omega^*, T\Omega^*\right)\right\}.$$

Again, since $\theta\left(\Omega_k, T\Omega^*\right) \leq s\theta\left(\Omega_k, \Omega^*\right) + s\theta\left(\Omega^*, T\Omega^*\right)$ for $k = n+1$ and $n+3$ respectively, we obtain that

$$M_T\left(\Omega_n, \Omega^*\right) \leq \max\left\{\theta\left(\Omega_{n+1}, \Omega^*\right), \theta\left(\Omega_{n+3}, \Omega^*\right), \frac{\theta\left(\Omega_{n+1}, \Omega^*\right) + \theta\left(\Omega^*, T\Omega^*\right)}{2}, \right.$$
$$\frac{\theta\left(\Omega_{n+3}, \Omega_{n+1}\right) + s\theta\left(\Omega_{n+3}, \Omega^*\right) + s\theta\left(\Omega^*, T\Omega^*\right)}{2s}, s\theta\left(\Omega_{n+3}, \Omega^*\right) + s\theta\left(\Omega^*, T\Omega^*\right) + \theta\left(\Omega_{n+3}, \Omega_{n+2}\right),$$
$$\left. s\theta\left(\Omega_{n+3}, \Omega^*\right) + s\theta\left(\Omega^*, T\Omega^*\right) + \theta\left(\Omega_{n+2}, \Omega_{n+1}\right), \theta\left(\Omega_{n+2}, \Omega^*\right) + \theta\left(\Omega^*, T\Omega^*\right)\right\}$$
$$\to \max\left\{0, \frac{\theta\left(\Omega^*, T\Omega^*\right)}{2}, s\theta\left(\Omega^*, T\Omega^*\right), \theta\left(\Omega^*, T\Omega^*\right)\right\} = s\theta\left(\Omega^*, T\Omega^*\right) \text{ when } n \to +\infty. \tag{20}$$

For the points $\Omega^*, T\Omega^*$, the following two inequalities are obvious:

$$\frac{1}{s}\theta\left(\Omega^*, T\Omega^*\right) \leq \theta\left(\Omega^*, \Omega_{n+1}\right) + \frac{1}{s^m} M_T\left(\Omega_n, \Omega^*\right) \text{ or } \frac{1}{s}\theta\left(\Omega^*, T\Omega^*\right) \leq \theta\left(\Omega^*, \Omega_{n+2}\right) + \frac{1}{s^m} M_T\left(\Omega_{n+1}, \Omega^*\right). \tag{21}$$

If $m \in (2, 5]$, then by (17)–(21), it follows that $\left(\frac{1}{s} - \frac{1}{s^{m-1}}\right)\theta\left(\Omega^*, T\Omega^*\right) \leq 0$ when $n \to +\infty$, that is, Ω^* is the unique fixed point of T. □

Remark 1. *From the proof of Theorem 6, it follows that it is true for all $m > 2$. Also, it is obvious that both functions F and ψ are superfluous in Theorem 6. This shows that our approach generalizes Theorem 4 (Theorem 2.2, [45]) in several directions.*

The following result is immediately a consequence of Theorem 6.

Corollary 3. *Let $(X, d, s > 1)$ be a b-complete b-metric space and $T : X \to X$ be a generalized $F-$Suzuki-contraction such that for all $\Omega, \omega \in X$ with $\theta\left(T\Omega, T\omega\right) > 0$ and $\frac{1}{2s}\theta\left(\Omega, T\Omega\right) < \theta\left(\Omega, \omega\right)$ implies:*

$$s^m\theta\left(T\Omega, T\omega\right) \leq \frac{1}{2} M_T\left(\Omega, \omega\right)$$
$$s^m\theta\left(T\Omega, T\omega\right) \leq M_T\left(\Omega, \omega\right)$$
$$\exp\left(s^m\theta\left(T\Omega, T\omega\right)\right) \leq \exp\left(M_T\left(\Omega, \omega\right)\right)$$
$$\exp\left(-s^m\theta\left(T\Omega, T\omega\right)\right) - \exp\left(s^m\theta\left(T\Omega, T\omega\right)\right) \leq \frac{1}{\exp\left(-M_T\left(\Omega, \omega\right)\right) - \exp\left(M_T\left(\Omega, \omega\right)\right)},$$

where $M_T\left(\Omega, \omega\right)$ is given as in Definition 6 and $m > 2$. Then in above cases, T has a unique fixed point Ω^* and for all $\Omega \in X$, the sequence $\{T^n\Omega\}_{n=1}^{+\infty}$ converges to Ω^*.

Proof. Because each of the following functions $F : (0, +\infty) \to (-\infty, +\infty)$

$$F\left(r\right) = r,\ \psi\left(r\right) = \frac{1}{2}r;\ F\left(r\right) = r,\ F\left(r\right) = \exp\left(r\right),\ F\left(r\right) = \exp\left(-r\right) - \exp\left(r\right),$$

are strictly increasing on $(0, +\infty)$, then the proof follows by Theorem 6. In the last three cases, the function ψ is not necessary. □

Remark 2. *It is worth to notice that some things in Reference [45] are doubt. For example:*

(1) The proof that $\lim_{n\to+\infty} M_T(\Omega_n, \Omega^*) = \theta(\Omega^*, T\Omega^*)$ is not correct (see page 9, line 8+), because b-metric d is not necessarily continuous. Namely, we obtained that $\lim_{n\to+\infty} M_T(\Omega_n, \Omega^*) = s\theta(\Omega^*, T\Omega^*)$, if $s > 1$ and $m \in (2, +\infty)$.

(2) Example 2.9 on pages 10–13 does not satisfy the assumptions of Theorem 2.2, that is, (Theorem 6). Indeed, since $X = \{-2, -1, 0, 1, 2\}$ and define the b-metric θ on it as

$$\theta(\Omega, \omega) = \begin{cases} 0, \text{ if } \Omega = \omega, \\ 4, \text{ if } (\Omega, \omega) \in \{(1, -1), (-1, 1)\}, \\ 1, \text{ otherwise.} \end{cases}$$

We have that $(X, \theta, 2)$ is a b-metric space. Let $T : X \to X$ be defined by

$$T(-2) = T(0) = T(2) = 0, T(-1) = 1, T(1) = -2.$$

Putting in (4) of Reference (p. 4, [45]) $\Omega = 0$ and $\omega = 1$, we get

$$\frac{1}{2\cdot 2}\theta(0, T0) < \theta(0, 1) \Rightarrow F\left(2^5\theta(T0, T1)\right) \le F(M_T(0, 1)) - \psi(M_T(0, 1)). \tag{22}$$

Because $\frac{1}{2\cdot 2}\theta(0, T0) < \theta(0, 1)$, that is, $0 < 1$ is true and since $2^5\theta(T0, T1) = 32\theta(0, -2) = 32$ and $M_T(0, 1) = 2$, the condition (22) becomes

$$0 < 1 \Rightarrow F(32) \le F(2) - \psi(2) < F(2).$$

According to condition (F1), it follows that $32 < 2$, which is a contradiction.

Author Contributions: Investigation, E.G., D.D.-Đ., H.A., Z.D.M. and D.P.; Methodology, D.D.-Đ.; Software, H.A., Z.D.M.; Supervision, Z.D.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Alsulami, H.H.; Karapinar, E.; Piri, H. Fixed points of generalized F-Suzuki type contraction in complete b-metric spaces. *Discret. Dyn. Nat. Soc.* **2015**, *2015*, 969726. [CrossRef]
2. Aydi, H.; Bota, M.F.; Karapinar, E.; Mitrović, S. A fixed point theorem for set-valued quasicontractions in b-metric spaces. *Fixed Point Theory Appl.* **2012**, *2012*, 88. [CrossRef]
3. Ding, H.-S.; Imdad, M.; Radenović, S.; Vujaković, J. On some fixed point results in b-metric, rectangular and b-rectangular metric spaces. *Arab J. Math. Sci.* **2016**, *22*, 151–164.
4. Qawaqneh, H.; Noorani, M.S.M.; Shatanawi, W.; Aydi, H.; Alsamir, H. Fixed point results for multi-valued contractions in b-metric spaces and an application. *Mathematics* **2018**, *7*, 132. [CrossRef]
5. Karapinar, E.; Czerwik, S.; Aydi, H. (α, ψ)-Meir-Keeler contraction mappings in generalized b-metric spaces. *J. Funct. Spaces* **2018**, *2018*, 3264620. [CrossRef]
6. Aydi, H.; Felhi, A.; Karapinar, E.; Sahmim, S. A Nadler-type fixed point theorem in dislocated spaces and applications. *Miscolc Math. Notes* **2018**, *19*, 111–124. [CrossRef]
7. Abdeljawad, T.; Mlaiki, N.; Aydi, H.; Souayah, N. Double controlled metric type spaces and some fixed point results. *Mathematics* **2018**, *6*, 320. [CrossRef]
8. Fabiano, N.; Došenović, T.; Rakić, D.; Radenović, S. Some new results on (s,q)- Dass-Gupta-Jaggi type contractive mappings in b-metric-like spaces. *Filomat* **2020**, in press.

9. Alamgir, N.; Kiran, Q.; Isik, H.; Aydi, H. Fixed point results via a Hausdorff controlled type metric. *Adv. Differ.* **2020**, *2020*, 24. [CrossRef]
10. Geraghty, M. On contractive mappings. *Proc. Am. Math. Soc.* **1973**, *40*, 604–608. [CrossRef]
11. Hammad, H.A.; De la Sen, M. Fixed-point results for a generalized almost (s, q)-Jaggi F-contraction-type on b-metric-like spaces. *Mathematics* **2020**, 8, 63. [CrossRef]
12. Aydi, H.; Lakzian, H.; Mitrović, Z.D.; Radenović, S. Best proximity points of MF-cyclic contractions with property UC. *Numer. Funct. Anal. Optim.* **2020**. [CrossRef]
13. Harandi, A.A. Metric-like spaces, partial metric spaces and fixed points. *Fixed Point Theory Appl.* **2012**, *2012*, 204. [CrossRef]
14. Mahmood, Q.; Shahzad, A.; Shoab, A.; Ansari, A.H.; Radenović, S. Common fixed point results for α-ψ-contractive mappings via (F, h) pair of upper class functions. *J. Math. Anal.* **2019**, *10*, 1–10.
15. Mitrović, Z.D.; Radenović, S. The Banach and Reich contractions in b_v (s)-metric spaces. *J. Fixed Point Theory Appl.* **2017**, *19*, 3087–3095. [CrossRef]
16. Radenović, S. Classical fixed point results in 0-complete partial metric spaces via cyclic-type extension. *Bull. Allahabad Math. Soc.* **2016**, *31*, 39–55.
17. De la Sen, M.; Nikolić, N.; Došenović, T.; Pavlović, M.; Radenović, S. Some results on $(s$-$q)$ -graphic contraction mappings in b-metric-like spaces. *Mathematics* **2019**, *7*, 1190. [CrossRef]
18. Parvaneh, V.; Haddadi, M.R.; Aydi, H. On Best Proximity Point Results for Some Type of Mappings. *J. Funct. Spaces* **2020**, *2020*, 6298138. [CrossRef]
19. Shukla, S.; Radenović, S.; Rajić, V.Ć. Some common fixed point theorems in 0-σ-complete metric-like spaces. *Vietnam J. Math.* **2013**, *41*, 341–352. [CrossRef]
20. Vujaković, J.; Aydi, H.; Radenović, S. Some remarks and new results in ordered partial b-metric spaces. *Mathematics* **2019**, *7*, 334. [CrossRef]
21. Younis, M.; Singh, D.; Radenović, S.; Imdad, M. Convergence theorems via generalized contractions and its applications. *Filomat* **2020**, in press.
22. Afshari, H.; Aydi, H.; Karapınar, E. On generalized α-ψ-Geraghty contractions on b-metric spaces. *Georgian Math. J.* **2020**, *27*, 9–21. [CrossRef]
23. Yu, D.; Chen, C.; Wang, H. Common fixed point theorems for $(T, g)_F$-contraction in b-metric-like spaces. *J. Inequal. Appl.* **2018**, *2018*, 222. [CrossRef] [PubMed]
24. Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundam. Math.* **1922**, *3*, 133–181. [CrossRef]
25. Ćirić, L.J. *Some Recent Results in Metrical Fixed Point Theory*; University of Belgrade: Beograd, Serbia, 2003.
26. Kirk, W.A.; Shahzad, N. *Fixed Point Theory in Distance Spaces*; Springer International Publishing: Cham, Switzerland, 2014.
27. Mlaiki, N.; Aydi, H.; Souayah, N.; Abdeljawad, T. Controlled metric type spaces and the related contraction principle. *Mathematics* **2018**, *6*, 194. [CrossRef]
28. Agarwal, R.P.; Karapinar, E.; O'Regan, D.; Roldan-Lopez-de-Hierro, A.F. *Fixed Point Theory in Metric Type Spaces*; Springer International Publishing: Cham, Switzerland, 2015.
29. Ameer, E.; Aydi, H.; Arshad, M.; De la Sen, M. Hybrid Ćirić type graphic (Υ, Λ)-contraction mappings with applications to electric circuit and fractional fifferential equations. *Symmetry* **2020**, *12*, 467. [CrossRef]
30. Aleksić, S.; Kadelburg, Z.; Mitrović, Z.D.; Radenović, S. A new survey: Cone metric spaces. *J. Int. Math. Virtual Inst.* **2019**, *9*, 93–121.
31. Frechet, M. Sur quelques points du calcul fonctionnel. *Rend. Circ. Mat. Palermo* **1906**, *22*, 1–74. [CrossRef]
32. Bakhtin, I.A. The contraction mapping principle in quasimetric spaces. *Funct. Anal. Ulianowsk Gos. Ped. Inst.* **1989**, *3*, 26–37.
33. Czerwik, S. Contraction mappings in b-metric spaces. *Acta Math. Inform. Univ. Ostrav.* **1993**, *1*, 5–11.
34. Shukla, S. Partial b-metric spaces and fixed point theorems. *Mediterr. J. Math.* **2014**, *11*, 703–711. [CrossRef]
35. Matthews, S.G. Partial metric topology, general topology and its applications. *Ann. N. Y. Acad. Sci.* **1994**, *728*, 183–197. [CrossRef]
36. Aleksić, S.; Mitrović, Z.D.; Radenović, S. Picard sequences in b-metric spaces. *Fixed Point Theory* **2020**, *21*, 35–46. [CrossRef]
37. Miculescu, R.; Mihail, A. New fixed point theorems for set-valued contractions in b-metric spaces. *J. Fixed Point Theory Appl.* **2017**, *19*, 2153–2163. [CrossRef]

38. Suzuki, T. Basic inequality on a b-metric space and its applications. *J. Inequal. Appl.* **2017**, *2017*, 256. [CrossRef]
39. Mitrović, Z.D. A note on the result of Suzuki, Miculescku and Mihail. *J. Fixed Point Theory Appl.* **2019**, *21*. [CrossRef]
40. Wardowski, D. Fixed point theory of a new type of contractive mappings in complete metric spaces. *Fixed Point Theory Appl.* **2012**, *2012*, 94. [CrossRef]
41. Wardowski, D.; Dung, N.V. Fixed points of F-weak contractions on complete metric spaces. *Demonstr. Math.* **2014**, *1*, 146–155. [CrossRef]
42. Collaco, P.; Silva, J.C. A complete comparison of 25 contraction conditions. *Nonlinear Anal. TMA* **1997**, *30*, 471–476. [CrossRef]
43. Rhoades, B.E. A comparison of various definitions of contractive mappings. *Trans. Am. Math. Soc.* **1977**, *226*, 257–290. [CrossRef]
44. Dung, N.V.; Hang, V.I. A fixed point theorem for generalized F-contraction on complete metric spaces. *Vietnam J. Math.* **2015**, *43*, 743–753. [CrossRef]
45. Piri, H.; Kumam, P. Fixed point theorems for generalized F-Suzuki-contraction mappings in complete b-metric spaces. *Fixed Point Theory Appl.* **2016**, *2016*, 90. [CrossRef]