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Remarks on the upper bound for the Randić energy of bipartite graphs

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ABSTRACT

Let $G = (V, E)$, $V = \{1, 2, \ldots, n\}$ be a simple graph without isolated vertices, with $n(n \geq 3)$ vertices and m edges, whose vertex degrees are given in the following form $d_1 \geq d_2 \geq \cdots \geq d_n > 0$. If A is the adjacency matrix, the Randić matrix $R = \|R_{ij}\|$ is defined in the following way

$$R_{ij} = \begin{cases} \dfrac{1}{\sqrt{d_i d_j}} & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of matrix R, $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$, are called the Randić eigenvalues of graph G. The Randić energy of graph G, denoted by RE, is defined in the following way:

$$RE = RE(G) = \sum_{i=1}^{n} |\rho_i|.$$

In this paper, upper bounds for graph invariant RE have been studied.

1. Introduction

Let G be a simple connected graph on the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. For $v_i \in V$, the degree of the vertex v_i, denoted by d_i, is the number of the vertices adjacent to v_i. Let A be the adjacency matrix of G and D is the diagonal matrix, whose diagonal elements are vertex degrees.

The Randić matrix of G is the $n \times n$ matrix $R = \|R_{ij}\|$ which is defined in the following way

$$R_{ij} = \begin{cases} \dfrac{1}{\sqrt{d_i d_j}} & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of matrix A [5] and $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ the eigenvalues of matrix R, which are called the Randić eigenvalues.

The (ordinary) energy $E(G)$ of graph G is defined as the sum of the absolute values of its eigenvalues:

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

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Details and more information on graph energy can be found in [13,14,11,10,22,16].

The Randić energy of graph G, denoted by RE, is graph invariant defined as

$$RE = RE(G) = \sum_{i=1}^{n} |\rho_i|.$$

The Randić energy is nowadays is an active field of research in applied mathematics. For its basic properties see the recent papers [1,7–9,12,15,17,21] and the references quoted therein.

In addition, let us remember that the general Randić index R_{-1} is defined in the following way [18]:

$$R_{-1} = \sum_{v_i v_j \in E(G)} \frac{1}{d_i d_j}.$$

2. Preliminary lemmas

Firstly, we give some known results about upper bounds for the Randić energy in terms of number of vertices n, number of edges m, minimum and maximum vertex degree and the general Randić index.

Lemma 2.1 ([3]). *Let G be a connected graph of order n. Then*

$$RE \le \rho_1 + \sqrt{(n-1)(2R_{-1} - \rho_1^2)}.$$

Lemma 2.2 ([19]). *Let G be a connected graph of order n. Then*

$$RE \le 1 + \sqrt{(n-1)(2R_{-1} - 1)}. \tag{1}$$

Lemma 2.3 ([2]). *Equality in (1) holds if and only if G is a complete graph or a non-bipartite connected graph with three distinct Randić eigenvalues*

$$\left(1, \sqrt{\frac{2R_{-1}-1}{n-1}}, -\sqrt{\frac{2R_{-1}-1}{n-1}}\right).$$

Lemma 2.4 ([23]). *Let G be a simple graph of order $n \ge 2$ with m edges. Then, for any real k_1 and k_2 with the properties $1 \le k_1 \le \sqrt{2R_{-1}/n}$ and $\sqrt{2R_{-1}/n} \ge k_2 \ge |\rho_n^*|$, where ρ_n^* is the smallest Randić eigenvalue with respect to absolute value,*

$$RE \le \min\left\{k_1 + \sqrt{(n-1)(2R_{-1} - k_1^2)},\ k_2 + \sqrt{(n-1)(2R_{-1} - k_2^2)},\ \sqrt{2nR_{-1} - \frac{n}{2}(1 - |\rho_n^*|)^2}\right\}. \tag{2}$$

Equality in (2) holds if and only if $G \cong K_n$.

Lemma 2.5 ([6]). *Let G be a simple graph of order n with minimal vertex degree δ. Then*

$$RE \le 1 + \sqrt{\frac{(n-1)(n-\delta)}{\delta}}.$$

3. Main result

Let G be a bipartite connected graph with n ($n \ge 3$) vertices, m edges, and the Randić eigenvalues $\rho_1 = 1 \ge \rho_1 \ge \cdots \ge \rho_{n-1} \ge \rho_n = -1$. Let $\rho = \max_{2 \le i \le n-1}\{|\rho_i|\}$.

Theorem 3.1. *Let G be a bipartite connected graph with n ($n \ge 3$) vertices and m edges. Then for every real number k, for which is $\rho \ge k \ge \sqrt{\frac{2(R_{-1}-1)}{n-2}}$, the following inequality holds*

$$RE \le 2 + k + \sqrt{(n-3)(2R_{-1} - 2 - k^2)}. \tag{3}$$

Equality in (3) holds if G is a complete bipartite graph, in which case $k = 0$.

Proof. In [20], a class $\mathcal{P}_n(a_1, a_2)$ of real polynomials of the form

$$P_n(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + b_3x^{n-3} + \cdots + b_n,$$

was considered, where $a_1,\ a_2$ are fixed real numbers. It was proven that for the roots $x_1 \geq x_2 \geq \cdots \geq x_n$ of that class of polynomials, the following inequality is valid

$$x_1 \leq \bar{x} + \frac{1}{n}\sqrt{(n-1)\Delta}, \tag{4}$$

where

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{and} \quad \Delta = n\sum_{i=1}^{n} x_i^2 - \Big(\sum_{i=1}^{n} x_i\Big)^2. \tag{5}$$

Now, consider the real polynomial

$$\varphi_n(x) = (x-1)^2\prod_{i=2}^{n-1}\big(x - |\rho_i|\big) = (x-1)^2(x^{n-2} + a_1x^{n-3} + a_2x^{n-4} + b_1x^{n-5} + \cdots + b_{n-3}),$$

and $a_1 = -\sum_{i=2}^{n-1}|\rho_i| = -(RE-2),\ a_2 = \frac{1}{2}\Big(\big(\sum_{i=2}^{n-1}|\rho_i|\big)^2 - \sum_{i=2}^{n-1}|\rho_i|^2\Big) = \frac{1}{2}\Big((RE-2)^2 - (2R_{-1}-2)\Big)$.

This polynomial belongs to a class of real polynomial $\mathcal{P}_n\big(2 - RE,\ \frac{1}{2}(RE-2)^2 - R_{-1} + 1\big)$.

From (5), for $n := n-2,\ x_i = |\rho_i|,\quad i = 2, \ldots, n-1$, we have

$$\bar{x} = \frac{1}{n-2}\sum_{i=2}^{n-1}|\rho_i| = \frac{RE-2}{n-2},$$

and

$$\Delta = (n-2)\sum_{i=2}^{n-1}|\rho_i|^2 - \Big(\sum_{i=2}^{n-1}|\rho_i|\Big)^2 = 2(n-2)(R_{-1}-1) - (RE-2)^2.$$

Then for every $k,\ \rho \geq k \geq \sqrt{\frac{2(R_{-1}-1)}{n-2}}$, from (4) the following inequalities hold

$$k \leq \rho \leq \frac{RE-2}{n-2} + \frac{1}{n-2}\sqrt{(n-3)(2(n-2)(R_{-1}) - (RE-2)^2)},$$

$$(n-2)k - (RE-2) \leq \sqrt{(n-3)(2(n-2)(R_{-1}) - (RE-2)^2)}. \tag{6}$$

Now, from given condition of the theorem,

$$(n-2)k - (RE-2) \geq \sqrt{2(n-2)(R_{-1}-1)} - (RE-2) \geq 0,$$

(see, for example [20]) and from (6) we have

$$(n-2)^2k^2 - 2(n-2)k(RE-2) + (RE-2)^2 \leq 2(n-2)(n-3)(R_{-1}-1) - (n-3)(RE-2)^2,$$

$$(n-2)k^2 - 2k(RE-2) + (RE-2)^2 \leq 2(n-3)(R_{-1}-1),$$

$$\Big((RE-2) - k\Big)^2 \leq (n-3)\Big(2(R_{-1}-1) - k^2\Big).$$

Now, inequality in (3) directly follows.

If G is a complete bipartite graph then $R_{-1}(G) = 1$, $\rho = 0$. It follows $k = 0$, $RE = 2$ and equality holds in (3). □

Corollary 3.1. *Let G be a connected bipartite graph with n, $n \geq 3$, vertices and m edges. Then*

$$RE \leq 2 + \rho + \sqrt{(n-3)(2R_{-1} - 2 - \rho^2)}, \tag{7}$$

with equality if G is a complete bipartite graph.

Corollary 3.2 ([2]). *Let G be a connected bipartite graph with n, $n \geq 3$, vertices and m edges. Then*

$$RE \leq 2 + \sqrt{2(n-2)(R_{-1}-1)}, \tag{8}$$

with equality if and only if G is a complete bipartite graph.

Proof. For $k = \sqrt{\frac{2(R_{-1}-1)}{n-2}}$ by (3) we have

$$\begin{aligned} RE &\le 2 + \sqrt{\frac{2(R_{-1}-1)}{n-2}} + \sqrt{(n-3)2(R_{-1}-1) - \frac{2(R_{-1}-1)}{n-2}} \\ &= 2 + \sqrt{\frac{2(R_{-1}-1)}{n-2}} + (n-3)\sqrt{\frac{2(R_{-1}-1)}{n-2}} = 2 + \sqrt{2(n-2)(R_{-1}-1)}. \end{aligned}$$

Remark 3.1. Upper bound (3) is better than upper bound (7) for $k > \frac{2(R_{-1}-1)}{n-2}$.

Remark 3.2. In [4] for the general Randić index, inequality $R_{-1} \le \frac{n}{2d_n}$ was proved. By this inequality and (8) we obtain

$$RE \le 2 + \sqrt{\frac{(n-2)(n-2d_n)}{d_n}},$$

where d_n denotes the smallest vertex degree.

This inequality was proved in [4].

From [18] we have the following result for the general Randić index: $R_{-1} \le \lfloor \frac{n}{2} \rfloor$. By this inequality and (8) we have

$$RE \le \begin{cases} n, & \text{if } n \text{ is even}, \\ 2 + \sqrt{(n-2)(n-3)}, & \text{if } n \text{ is odd}. \end{cases}$$

Equality holds if and only if $G = K_2$.

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