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Article

Internal Variable Theory in Viscoelasticity: Fractional Generalizations and Thermodynamical Restrictions

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Abstract: Here, we study the internal variable approach to viscoelasticity. First, we generalize the classical approach by introducing a fractional derivative into the equation for time evolution of the internal variables. Next, we derive restrictions on the coefficients that follow from the dissipation inequality (entropy inequality under isothermal conditions). In the example of wave propagation, we show that the restrictions that follow from entropy inequality are sufficient to guarantee the existence of the solution. We present a numerical solution to the wave equation for several values of the parameters.

Keywords: fractional calculus; internal variables; thermodynamical admissibility

MSC: 26A33; 74S40

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1. Introduction

The internal variable method represents the common procedure for studying the constitutive equations, both linear and nonlinear, of viscoelastic materials. It has importance for both the analytical and numerical aspects (see [1,2]). In [3], we analyzed the internal variable approach to viscoelasticity for a single internal variable that is described by a fractional evolution equation. We showed that the thermodynamical stability condition imposes restrictions on the coefficients in the constitutive equation that, when the internal variable is eliminated, agree with the earlier obtained results.

Our aim in this work is to analyze the more general case of a material with several internal variables. The evolution of the internal variables is assumed to be described by a linear system of fractional differential equations of different orders, as in [4]. For the analysis of the restrictions that follow from the second law of thermodynamics under isothermal conditions, we shall apply the method used in [5,6]. Thus, our model is the same as the one analyzed in [7]. However, our analysis is different, and the restrictions that we obtain are more general. We shall analyze two specific models. Application of the fractional calculus and the results presented here is possible in various areas of physics and mechanics. The experimental results that are needed to determine the order of the fractional derivatives as well as the coefficients in the constitutive equation may be obtained from experiments, such as those in [8,9].

2. Model

We follow [1] in presenting the basics of the internal variable method in linear viscoelasticity. Thus, we consider a one-dimensional viscoelastic body whose constitutive equation is given in the form of a generalized standard linear model (also called the generalized Maxwell model), Wiechert model, or Maxwell–Wiechert model. This model is equivalent to the following (see [1] p. 35, [10]):

$$\sigma(t,x) = k\mathcal{E}(t,x) + \sum_{j=1}^{N} k_j \varepsilon_j(t,x), \tag{1}$$

where

$$\frac{d\varepsilon_j}{dt} + \frac{1}{\tau_j}\varepsilon_j(t,x) = \frac{d\mathcal{E}}{dt}, \quad \varepsilon_j(0,x) = 0, \quad j = 1,\ldots,N. \tag{2}$$

Here, σ denotes the Cauchy stress at the point x, where $x \in (-\infty,\infty)$ denotes a spatial coordinate and $t \in (0,\infty)$ is the time. In addition, $\mathcal{E}$ denotes the strain and ε_j, $j = 1,\ldots,N$ denote the internal variables. We note that Equations (1) and (2) are of the forms in Equations (11) and (12) of [11]. Here, $\mathcal{E}$ is the observable variable and ε_j represents the hidden or internal variables.

In this work, we propose to generalize the system in Equations (1) and (2) by introducing fractional derivatives. Since fractional derivatives are nonlocal operators, the generalization that we propose amounts to the introduction of memory effects in the constitutive equation. Recall that the Riemann–Liouville fractional derivative of the real order $\alpha \in [0,1]$ is defined as [12]

$${}_0D_t^{\alpha} f(t) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}} d\tau, \;\; t > 0, \tag{3}$$

for absolutely continuous functions $f(t)$, $t \in [0,T]$ $(f \in AC([0,T]))$. In the analysis that follows, we assume that Equation (1) holds, but instead of Equation (2), we have

$${}_0D_t^{\alpha_j} \varepsilon_j(t,x) + \frac{1}{\tau_j}\varepsilon_j(t,x) = {}_0D_t^{\beta} \mathcal{E}(t,x), \quad \varepsilon_j(0,x) = 0,$$

where we assume that

$$0 < \alpha_j, \beta \le 1, \quad j = 1,\ldots,N. \tag{4}$$

Recently, in [13], *general fractional derivatives* were studied in detail. In the approach in [13], the Riemann–Liouville fractional derivative is just a special case. In principle, general fractional derivatives may be used instead of Equation (3).

Our intention here is to derive the restrictions on the coefficients in Equations (1) and (4), which follow from the second law of thermodynamics under isothermal conditions. We assume that $\varepsilon_j(t,x) = 0$, $j = 1,\ldots,N$, $\mathcal{E}(t,x) = 0$ and $\sigma(t,x) = 0$ for each $x \in \mathbb{R}$ and $t < 0$. Furthermore, assume that $\mathcal{E} \in C^1([0,\infty))$. The second law of thermodynamics, under isothermal conditions, requires that for any cycle of duration $T > 0$, with "cycle" here meaning $\mathcal{E}(0) = \mathcal{E}(T) = 0$, there exists $D > 0$ such that the dissipation inequality is

$$D(x) = \int_0^T \sigma(t,x)\mathcal{E}^{(1)}(t,x)dt \ge 0, \tag{5}$$

where $\mathcal{E}^{(1)}(t,x) = \frac{\partial \mathcal{E}(t,x)}{\partial t}$ holds for every $x \in (-\infty,\infty)$. The inequality in Equation (5) is used for any value of $\mathcal{E}$ that does not necessarily satisfy the conditions of a cycle. For example, in [14], it is proposed to use Equation (5) for $\mathcal{E}$ satisfying "any sufficiently smooth $\mathcal{E}$" which satisfies $\mathcal{E}(x,t) = 0$, $t \in (-\infty,0]$. We shall use this requirement, since it does not require the definition of a cycle, as cycles can be defined differently. See, for example, ref. [15], where it is required that the entropy inequality—and Equation (5) is just special case of it—holds for specially defined D-cyclic processes. More details on the dissipativity condition may be found in [16–18]. Since Equation (5) must hold for all x, in the analysis that follows, we omit x as an independent variable. By applying the Fourier transform

$\hat{\sigma}(\omega) = \mathcal{F}(\sigma)(\omega) = \int_{-\infty}^{\infty} \sigma(t)e^{-i\omega t}dt$, $\omega \in \mathbb{R}$, to Equations (1) and (2) and eliminating $\hat{\varepsilon}_j(\omega)$, we obtain

$$\hat{\sigma}(\omega) = E(\omega)\widehat{\mathcal{E}}(\omega), \tag{6}$$

where

$$E(\omega) = E_1(\omega) + iE_2(\omega) = \left[k + (i\omega)^{\beta} \sum_{j=1}^{N} \frac{k_j}{(i\omega)^{\alpha_j} + \frac{1}{\tau_j}}\right] \widehat{\mathcal{E}}(\omega).$$

We now recall the results of Proposition 2.4 in [5]:

Proposition 1. *A necessary condition for a constitutive equation to satisfy Equation (5) is that the components E_1 and E_2 of the complex dynamic modulus E, defined by Equation (6), satisfy*

$$\begin{aligned} E_1(\omega) &= E_1(-\omega), \\ E_2(\omega) &= -E_2(-\omega),\ \omega \in \mathbb{R},\ E_2(\omega) \geq 0,\ \omega \in \mathbb{R}_+, \\ \int_0^{\infty} \frac{1}{\omega} \frac{E_2(\omega)}{(1+\omega^2)^{\frac{m}{2}}} d\omega &< \infty, \textit{ for some } m > 0. \end{aligned} \tag{7}$$

Proof. Here, we sketch the proof of the proposition. Let $F_1(\omega) = E_2(\omega)/\omega$. Since $E_2(\omega)$ is odd, we will show that the non-negativity of $F_1(\omega) \geq 0$, $\omega \in (-\infty, \infty)$ implies that Equation (5) holds. Let $\kappa(t)$ be the characteristic function of $[0, T]$, and let $\theta \in \mathcal{S}(\mathbb{R})$, where $\mathcal{S}$ is the Schwartz space (i.e., the space of all smooth functions that are rapidly decreasing at infinity along with all derivatives). Then, by the Bochner theorem [19], the positivity of $\kappa(t)\kappa(\tau)\mathcal{F}^{-1}(F_1)(t-\tau)$ implies the following for every $\theta \in \mathcal{S}(\mathbb{R})$:

$$\int_0^T \int_0^T \kappa(t)\kappa(\tau)\mathcal{F}^{-1}(F_1(\xi))(t-\tau)\theta(\tau)\theta(t)d\tau dt =$$

$$= 2\int_0^T \int_0^t \mathcal{F}^{-1}(F_1(\xi))(t-\tau)\theta(\tau)\theta(t)d\tau dt \geq 0,$$

where we used $F_1(-\xi) = F_1(\xi)$, and in the last step, we used the Fubini theorem to obtain 2 in front of the integral. Finally, since any function in $C[0, T]$ supported by $[0, T]$ is a limit of a real-valued sequence θ_k, $k \in \mathbb{N}$, let $\theta_k \to \varepsilon'$, $\operatorname{supp} \theta_k \subseteq [0, T]$ such that $\theta_k \to \varepsilon^{(1)}$, $k \to \infty$ uniformly on $[0, T]$. For such θ_k, the following applies:

$$\lim_{k\to\infty} \int_0^T \int_0^T \kappa(t)\kappa(\tau)\mathcal{F}^{-1}(F_1(\xi))(t-\tau)\theta_k(\tau)\theta_k(t)d\tau dt =$$

$$= \int_0^T \int_0^T \mathcal{F}^{-1}(F_1(\xi))(t-\tau)\varepsilon^{(1)}(\tau)\varepsilon^{(1)}(t)d\tau dt \geq 0.$$

The last expression is just $D \geq 0$ (i.e., (5) holds). □

The condition in Equation $(7)_3$ guaranties that $\frac{E_2}{\omega}$ is a tempered measure (see [20]). The conditions in Equation (7) are different from the restrictions that follow from the Bagley–Torvik method (see [1,21]). The restrictions derived by using the Bagley–Torvik method are based on the assumption that under sinusoidal stress imposed on a viscoelastic body, after the transition period, the strain has the same form but with a phase shift. Then, the energy loss during a complete cycle is required to be positive. In the analysis leading to Equation (7), the energy dissipation condition is satisfied during the deformation process starting from the virginal state, not requiring that this deformation constitutes a cycle. The approach used here was also used in [17] (p. 128).

We conclude by stating that our goal in this work is to analyze the restriction and thermodynamical admissibility that follows from Equation (5) in the new constitutive equations:

(A) Fractional-order internal variable viscoelasticity with constitutive equation of the form that generalizes Equations (1) and (2) such that

$$\sigma(t,x) = k\mathcal{E}(t,x) + \sum_{j=1}^{N} k_j \varepsilon_j(t,x), \tag{8}$$

$${}_0D_t^{\alpha}\varepsilon_j(t,x) + \frac{1}{\tau_j}\varepsilon_j(t,x) = {}_0D_t^{\beta}\mathcal{E}(t,x), \quad \varepsilon_j(0,x) = 0. \tag{9}$$

The case where $\alpha_j = \beta = 0$, $j = 1,\ldots,N$, is trivial, since it leads to Hooke's law:

$$\sigma(t,x) = \left[k + \sum_{j=1}^{N} \frac{k_j}{1+\frac{1}{\tau_j}}\right]\mathcal{E}(t,x). \tag{10}$$

In the analysis that follows, we therefore assume that

$$k \geq 0, \quad \tau_j \geq 0, \tag{11}$$

while for $k_j, j = 1,\ldots,N$, we assume that it could be both positive and negative. The models in Equations (8) and (9) with $\alpha_j = 1$, $j = 1,\ldots,N$, are used to describe the mechanical behavior of foods (see [22]).

(B) Distributed fractional-order internal variable viscoelasticity with a constitutive equation of the form that generalizes the model presented in [1] (p. 36):

$$\sigma(t,x) = k\mathcal{E}(t,x) + \int_0^B \phi(\alpha)\varepsilon_\alpha(t,x)d\alpha, \tag{12}$$

$$\tau^{\alpha}\ {}_0D_t^{\alpha}\varepsilon_\alpha(t,x) + \varepsilon_\alpha(t,x) = \tau_1^{\beta}\ {}_0D_t^{\beta}\mathcal{E}(t,x), \quad \varepsilon_\alpha(0,x) = 0, \tag{13}$$

where $\phi(\alpha)$, $\alpha \in [0,B]$ is given a constitutive function or distribution. In addition, τ and τ_1 denote the relaxation times. In the analysis that follows, we shall assume that $\phi = \mu = const$. Note that Equations (12) and (13) are fractional generalizations of the internal variable model with a continuum of relaxation times, as presented in [1] (p. 36). They also have the form of a distributed-order fractional constitutive equations.

3. Main Results

We treat cases A and B separately.

3.1. Model A

In order to analyze the restrictions which follow from Equation (5), we apply the Fourier transform $\hat{\sigma}(\omega) = \mathcal{F}(\sigma)(\omega) = \int_{-\infty}^{\infty}\sigma(t)e^{-i\omega t}dt$, $\omega \in \mathbb{R}$, to Equations (8) and (9) and obtain

$$\begin{aligned} \widehat{\sigma}(\omega) &= \left[k + \sum_{j=1}^{N} k_j\widehat{\varepsilon}_j(\omega)\right]\widehat{\mathcal{E}}(\omega), \\ (i\omega)^{\beta}\widehat{\mathcal{E}}(\omega) &= \widehat{\varepsilon}_j(\omega)\left[(i\omega)^{\alpha_j} + \frac{1}{\tau_j}\right], \quad j = 1,\ldots,N. \end{aligned} \tag{14}$$

In Equation (14), we omitted the dependance on x. From Equation (14), we obtain

$$\widehat{\sigma}(\omega) = \left[k + (i\omega)^{\beta}\sum_{j=1}^{N} \frac{k_j}{(i\omega)^{\alpha_j} + \frac{1}{\tau_j}}\right]\widehat{\mathcal{E}}(\omega). \tag{15}$$

From Equation (15), we obtain the complex modulus E (cf. [23]) as

$$E(\omega) = E_1(\omega) + iE_2(\omega) = \left[k + (i\omega)^{\beta} \sum_{j=1}^{N} \frac{k_j}{(i\omega)^{\alpha_j} + \frac{1}{\tau_j}}\right], \quad \omega \in \mathbb{R}. \tag{16}$$

Note that the classical case in Equations (1) and (2) is recovered for $\alpha_j = \beta = 1$, $j = 1, \ldots, N$. Since $(i\omega)^{\alpha_j} = \omega^{\alpha_j}\left[\cos\frac{\pi\alpha_j}{2} + i\sin\frac{\pi\alpha_j}{2}\right]$, $\omega > 0$, we obtain

$$\begin{aligned} E_1(\omega) &= \Bigg[k + \omega^{\beta}\cos\frac{\pi\beta}{2}\sum_{j=1}^{N} k_j \frac{\omega^{\alpha_j}\cos\frac{\pi\alpha_j}{2} + \frac{1}{\tau_j}}{\left[\omega^{\alpha_j}\cos\frac{\pi\alpha_j}{2} + \frac{1}{\tau_j}\right]^2 + \left[\omega^{\alpha_j}\sin\frac{\pi\alpha_j}{2}\right]^2} + \\ &\quad + \omega^{\beta}\sin\frac{\pi\beta}{2}\sum_{j=1}^{N} k_j \frac{\omega^{\alpha_j}\sin\frac{\pi\alpha_j}{2}}{\left[\omega^{\alpha_j}\cos\frac{\pi\alpha_j}{2} + \frac{1}{\tau_j}\right]^2 + \left[\omega^{\alpha_j}\sin\frac{\pi\alpha_j}{2}\right]^2}\Bigg], \end{aligned} \tag{17}$$

$$\begin{aligned} E_2(\omega) &= \Bigg[\omega^{\beta}\sin\frac{\pi\beta}{2}\sum_{j=1}^{N} k_j \frac{\omega^{\alpha_j}\cos\frac{\pi\alpha_j}{2} + \frac{1}{\tau_j}}{\left[\omega^{\alpha_j}\cos\frac{\pi\alpha_j}{2} + \frac{1}{\tau_j}\right]^2 + \left[\omega^{\alpha_j}\sin\frac{\pi\alpha_j}{2}\right]^2} - \\ &\quad - \omega^{\beta}\cos\frac{\pi\beta}{2}\sum_{j=1}^{N} k_j \frac{\omega^{\alpha_j}\sin\frac{\pi\alpha_j}{2}}{\left[\omega^{\alpha_j}\cos\frac{\pi\alpha_j}{2} + \frac{1}{\tau_j}\right]^2 + \left[\omega^{\alpha_j}\sin\frac{\pi\alpha_j}{2}\right]^2}\Bigg], \end{aligned} \tag{18}$$

$$\omega > 0,$$

or

$$\begin{aligned} E_1(\omega) &= k + \sum_{j=1}^{N} k_j \frac{\left[\omega^{\beta+\alpha_j}\cos\frac{\pi(\beta-\alpha_j)}{2} + \omega^{\beta}\frac{1}{\tau_j}\cos\frac{\pi\beta}{2}\right]}{\left[\omega^{\alpha_j}\cos\frac{\pi\alpha_j}{2} + \frac{1}{\tau_j}\right]^2 + \left[\omega^{\alpha_j}\sin\frac{\pi\alpha_j}{2}\right]^2}, \\ E_2(\omega) &= \sum_{j=1}^{N} k_j \frac{\left[\omega^{\beta+\alpha_j}\sin\frac{\pi(\beta-\alpha_j)}{2} + \omega^{\beta}\frac{1}{\tau_j}\sin\frac{\pi\beta}{2}\right]}{\left[\omega^{\alpha_j}\cos\frac{\pi\alpha_j}{2} + \frac{1}{\tau_j}\right]^2 + \left[\omega^{\alpha_j}\sin\frac{\pi\alpha_j}{2}\right]^2}. \end{aligned}$$

For $\omega < 0$ in the above formulas, $\frac{\pi}{2}$ should be replaced with $-\frac{\pi}{2}$. From Equations (17) and (18), it follows that

$$E_1(\omega) = E_1(-\omega), \quad E_2(\omega) = -E_2(-\omega). \tag{19}$$

Our main result is stated as follows:

Theorem 1. *Sufficient conditions for Equations (8) and (9) to satisfy Equations (5) and (7) are*

$$k \geq 0, \quad \sum_{j=1}^{N} \frac{k_j}{1 + \frac{1}{\tau_j}} \geq 0, \quad \alpha_j \leq \beta, \quad j = 1, \ldots, N. \tag{20}$$

Proof. First, we consider the case $\alpha_j = \beta = 0$, $j = 1, \ldots, N$. In this case, Equations (8) and (9) define Hooke's body, so Equation (5) holds when Equation (13) is satisfied since the storage modulus E_1 is different from zero and the loss modulus E_2 is equal to zero. Then, from Equation (10), we get

$$\sigma(t,x) = E\mathcal{E}(t,x),$$

with $E = k + \sum\limits_{j=1}^{N} \frac{k_j}{1+\frac{1}{\tau_j}} > 0$ if Equation (20) holds. In the case of $0 < \alpha_j, \beta \le 1$, $j = 1, \ldots, N$, the conditions in Equation (19) are satisfied by Equations (17) and (18). We now consider the condition in Equation $(7)_1$ (i.e., $E_2(\omega) \ge 0$ for $\omega > 0$). Given $\omega^* > 0$, Equation (18) could be estimated as follows:

$$
\begin{aligned}
E_2(\omega) &= \sum_{j=1}^{N} k_j \frac{\left[\omega^{*(\beta+\alpha_j)} \sin\frac{\pi(\beta-\alpha_j)}{2} + \omega^{*\beta}\frac{1}{\tau_j}\sin\frac{\pi\beta}{2}\right]}{\left[\omega^{*\alpha_j}\cos\frac{\pi\alpha_j}{2} + \frac{1}{\tau_j}\right]^2 + \left[\omega^{*\alpha_j}\sin\frac{\pi\alpha_j}{2}\right]^2} \ge \\
&\ge \omega^{*\beta} \frac{\min\limits_j \left[\left(\omega^{*\alpha_j}\sin\frac{\pi(\beta-\alpha_j)}{2} + \frac{1}{\tau_j}\sin\frac{\pi\beta}{2}\right)\frac{k_j}{1+\frac{1}{\tau_j}}\right]}{\max\limits_j \left[\left(\left[\omega^{*\alpha_j}\cos\frac{\pi\alpha_j}{2} + \frac{1}{\tau_j}\right]^2 + \left[\omega^{*\alpha_j}\sin\frac{\pi\alpha_j}{2}\right]^2\right)\frac{1}{1+\frac{1}{\tau_j}}\right]} \cdot \sum_{j=1}^{N}\frac{k_j}{1+\frac{1}{\tau_j}}.
\end{aligned}
$$

Therefore, $E_2(\omega) \ge 0$ if $\beta \ge \alpha_j$, $j = 1, \ldots, N$, and Equation $(20)_2$ holds. Finally, the condition in Equation $(7)_2$ becomes

$$
\begin{aligned}
&\int_0^\infty \sum_{j=1}^{N} k_j \frac{\left[\omega^{\beta+\alpha_j}\sin\frac{\pi(\beta-\alpha_j)}{2} + \omega^{\beta}\frac{1}{\tau_j}\sin\frac{\pi\beta}{2}\right]}{\omega(1+\omega^2)^{\frac{m}{2}}\left[\omega^{\alpha_j}\cos\frac{\pi\alpha_j}{2} + \frac{1}{\tau_j}\right]^2 + \left[\omega^{\alpha_j}\sin\frac{\pi\alpha_j}{2}\right]^2} d\omega \le \\
\le &\int_0^\infty \sum_{j=1}^{N} k_j \frac{\left[\omega^{\beta+\alpha_j}\sin\frac{\pi(\beta-\alpha_j)}{2} + \omega^{\beta}\frac{1}{\tau_j}\sin\frac{\pi\beta}{2}\right]}{\omega(1+\omega^2)^{\frac{m}{2}}\frac{1}{\tau_j^2}} d\omega < \infty.
\end{aligned}
$$

This ends the proof. □

Remark 1. *The conditions in Theorem 1 generalize the results of [7] where, in our notation, $k_j > 0$, $j = 1, \ldots, N$ is required. Here, some of k_j, $j = 1, \ldots, N$ may be negative. This is enough that*

$$
\sum_{j=1}^{N} \frac{k_j}{1+\frac{1}{\tau_j}} \ge 0. \tag{21}
$$

In addition, for any $\omega^* > 0$, we have

$$
\begin{aligned}
E_1(\omega^*) &= k + \sum_{j=1}^{N} k_j \frac{\left[\omega^{*(\beta+\alpha_j)}\cos\frac{\pi(\beta-\alpha_j)}{2} + (\omega^*)^{\beta}\frac{1}{\tau_j}\cos\frac{\pi\beta}{2}\right]}{\left[(\omega^*)^{\alpha_j}\cos\frac{\pi\alpha_j}{2} + \frac{1}{\tau_j}\right]^2 + \left[(\omega^*)^{\alpha_j}\sin\frac{\pi\alpha_j}{2}\right]^2} \ge \\
&\ge k + \sum_{j=1}^{N} k_j \frac{(\omega^*)^{\beta}\min\limits_j\left[(\omega^*)^{\alpha_j}\cos\frac{\pi(\beta-\alpha_j)}{2} + \frac{1}{\tau_j}\cos\frac{\pi\beta}{2}\right]\frac{1}{1+\frac{1}{\tau_j}}}{\max\limits_j\left[\left\{\left[(\omega^*)^{\alpha_j}\cos\frac{\pi\alpha_j}{2} + \frac{1}{\tau_j}\right]^2 + \left[(\omega^*)^{\alpha_j}\sin\frac{\pi\alpha_j}{2}\right]^2\right\}\frac{1}{1+\frac{1}{\tau_j}}\right]} = \\
&= k + \frac{(\omega^*)^{\beta}\min\limits_j\left[(\omega^*)^{\alpha_j}\cos\frac{\pi(\beta-\alpha_j)}{2} + \frac{1}{\tau_j}\cos\frac{\pi\beta}{2}\right]}{\max\limits_j\left[\left\{\left[(\omega^*)^{\alpha_j}\cos\frac{\pi\alpha_j}{2} + \frac{1}{\tau_j}\right]^2 + \left[(\omega^*)^{\alpha_j}\sin\frac{\pi\alpha_j}{2}\right]^2\right\}\frac{1}{1+\frac{1}{\tau_j}}\right]} \cdot \sum_{j=1}^{N}\frac{k_j}{1+\frac{1}{\tau_j}} \ge k > 0.
\end{aligned}
$$

Therefore, $E_1(\omega) > 0$ if Equation (20) holds.

3.2. Model B

For case B, we follow the same procedure as in case A and obtain the following from Equations (12) and (13):

$$\widehat{\sigma}(\omega) = \left[k + \mu(\tau_1 i\omega)^\beta \int_0^B \frac{d\alpha}{1+(i\tau\omega)^\alpha}\right]\widehat{\mathcal{E}}(\omega), \quad \omega \in \mathbb{R}. \tag{22}$$

By using Equations (22) and (7), we have following theorem:

Theorem 2. *A sufficient condition such that Equations (12) and (13) with $\phi = \mu = const.$ satisfies Equation (5) is*

$$\mu > 0 \text{ and } B \leq \beta \leq 1. \tag{23}$$

Proof. We apply the Fourier transform to Equations (12) and (13) with $\phi = \mu$. Note that

$$\int_0^B \frac{d\alpha}{1+(i\tau\omega)^\alpha} = \int_0^B \frac{1+(\tau\omega)^\alpha \cos\frac{\alpha\pi}{2} - i(\tau\omega)^\alpha \sin\frac{\alpha\pi}{2}}{\left[1+(\tau\omega)^\alpha\cos\frac{\alpha\pi}{2}\right]^2 + \left[(\tau\omega)^\alpha \sin\frac{\alpha\pi}{2}\right]^2} d\alpha, \quad \omega > 0, \tag{24}$$

For $\omega < 0$, the term $\frac{\pi}{2}$ should be replaced with $\frac{-\pi}{2}$. By using Equation (24), we obtain the complex modulus $E(\omega) = E_1(\omega) + iE_2(\omega)$ as

$$\begin{aligned}
E_1(\omega) &= k + \mu \int_0^B \frac{\left(1+(\tau\omega)^\alpha\cos\frac{\alpha\pi}{2}\right)(\omega\tau_1)^\beta \cos\frac{\beta\pi}{2} + (\tau\omega)^\alpha(\omega\tau_1)^\beta \sin\frac{\alpha\pi}{2}\sin\frac{\beta\pi}{2}}{\left[1+(\tau\omega)^\alpha\cos\frac{\alpha\pi}{2}\right]^2 + \left[(\tau\omega)^\alpha\sin\frac{\alpha\pi}{2}\right]^2} d\alpha, \\
E_2(\omega) &= \mu \int_0^B \frac{(\omega\tau_1)^\beta \sin\frac{\beta\pi}{2} + (\tau\omega)^\alpha(\omega\tau_1)^\beta \sin\frac{\pi}{2}(\beta-\alpha)}{\left[1+(\tau\omega)^\alpha\cos\frac{\alpha\pi}{2}\right]^2 + \left[(\tau\omega)^\alpha\sin\frac{\alpha\pi}{2}\right]^2} d\alpha.
\end{aligned} \tag{25}$$

From Equation (25), we conclude that E_1 and E_2 satisfy all the conditions stated in Equation (7) if $\mu > 0$ and $B \leq \beta$. Therefore, Equation (7) is satisfied, and the result follows. □

4. Examples

We apply Theorem 1 in the two special cases of constitutive Equations (8) and (9).

- The generalized Zener model;
  Suppose that $N = 1$, $0 < \beta, \alpha_1 < 1$ so that

$$\begin{aligned}
\sigma(t,x) &= k\mathcal{E}(t,x) + k_1\varepsilon_1(t,x), \\
{}_0D_t^\alpha \varepsilon_1(t,x) + \frac{1}{\tau_1}\varepsilon_1(t,x) &= {}_0D_t^\beta \mathcal{E}(t,x), \quad \varepsilon_1(0,x) = 0.
\end{aligned} \tag{26}$$

The restrictions on the coefficients are Equations (7) and $(20)_{1,2}$:

$$k \geq 0, \quad \tau_1 > 0, \quad k_1 \geq 0, \quad \alpha \leq \beta. \tag{27}$$

By applying the operator ${}_0D_t^\alpha$ to Equation $(26)_1$ we obtain

$$\begin{aligned}
{}_0D_t^\alpha \sigma(t,x) &= k\,{}_0D_t^\alpha \mathcal{E}(t,x) + k_1\,{}_0D_t^\alpha \varepsilon_1(t,x) \\
&= k\,{}_0D_t^\alpha \mathcal{E}(t,x) + k_1\left[{}_0D_t^\beta \mathcal{E}(t,x) - \frac{1}{\tau_1}\varepsilon_1(t,x)\right] \\
&= k\,{}_0D_t^\alpha \mathcal{E}(t,x) + k_1\left[{}_0D_t^\beta \mathcal{E}(t,x) - \frac{1}{\tau_1}\left(\frac{\sigma(t,x) - k\mathcal{E}(t,x)}{k_1}\right)\right],
\end{aligned}$$

Alternatively, if $k \neq 0$, we have

$$\tau_1\,{}_0D_t^\alpha \sigma(t) + \sigma(t) = k\left[\mathcal{E}(t,x) + \tau_1 D_t^\alpha \mathcal{E}(t,x) + \frac{k_1}{k}\tau_{1 0}D_t^\beta \mathcal{E}(t,x)\right]. \tag{28}$$

Equation (28) is the generalized Zener model. For $\alpha = \beta$, we obtain the standard fractional Zener model:

$$A\,{}_0D_t^{\alpha}\sigma(t,x) + \sigma(t,x) = k[\mathcal{E}(t,x) + B\,{}_0D_t^{\alpha}\mathcal{E}(t,x)],$$

with the known restrictions

$$A = \tau_1 \leq B = \tau_1(1 + \frac{k_1}{k}).$$

In [3], this is also obtained as a special case of the internal variable model by a different procedure.

- Next, we treat the creep problem for the creep in model A (i.e., Equations (8) and (9) with $\alpha_1 = 0.2$, $\alpha_2 = 0.5$, $\alpha_3 = 0.9$, $\beta = 0.9$, $\tau_j = 1$, $j = 1,2,3$, and $k = 1$, $k_1 = 1$, $k_2 = (1,-1,-1.5)$, $k_3 = 1$). By application of the Laplace transform $\mathcal{L}(f)(s) = \int_0^{\infty} \exp(-ts)f(t)dt = \overline{f}(s)$ to Equations (8) and (9), we obtain

$$\overline{\sigma}(s,x) = E(s)\overline{\mathcal{E}}(s,x) = \left[k + s^{\beta}\sum_{j=1}^{3}\frac{k_j}{s^{\alpha_j} + \frac{1}{\tau_j}}\right]\overline{\mathcal{E}}(s,x).$$

For $\tau_j > 0$, $\alpha_j \in (0,1)$, there are no solutions to $s^{\alpha_j} + \frac{1}{\tau_j} = 0$ with a positive real part. Also $k > 0$ implies that, $1/E(s)$ has no singular points in $\mathbb{C}_+ = \{s \in \mathbb{C}\,|\,\mathrm{Re}\,s > 0\}$. When solving for $\mathcal{E}$, we have

$$\mathcal{E}(t,x) = \mathcal{L}^{-1}\left(\frac{\overline{\sigma}(s,x)}{E(s)}\right)(t) = \mathcal{L}^{-1}\left(\frac{\overline{\sigma}(s,x)}{k + s^{\beta}\sum\limits_{j=1}^{3}\frac{k_j}{s^{\alpha_j}+\frac{1}{\tau_j}}}\right)(t) = \frac{1}{2\pi i}\int_L \frac{\exp(ts)\overline{\sigma}(s,x)}{k + s^{\beta}\sum\limits_{j=1}^{3}\frac{k_j}{s^{\alpha_j}+\frac{1}{\tau_j}}}ds, \quad (29)$$

where $L = \{s : s = x_0 + ip,\ x_0 > 0,\ p \in (-\infty,\infty)\}$. For the creep test, we have $\sigma(t,x) = H(t)\sigma_0$, where H is a Heaviside step function and $\sigma_0 = const. > 0$. Since $\overline{H}(s) = \sigma_0/s$, the inversion of Equation (29) becomes

$$\mathcal{E}(t,x) = \sigma_0 \lim_{P\to\infty}\frac{1}{2\pi}\int_{-P}^{P}\frac{\exp(t(x_0+ip))}{(x_0+ip)\left[k + (x_0+ip)^{\beta}\sum\limits_{j=1}^{3}\frac{k_j}{(x_0+ip)^{\alpha_j}+\frac{1}{\tau_j}}\right]}dp. \quad (30)$$

The results of the numerical inversion of Equation (30) for $\sigma_0 = 1$ are shown in Figure 1. Also in Figure 1, we chose the values of the parameters shown in Table 1. In three cases the derived restrictions were satisfied and one where they were violated.

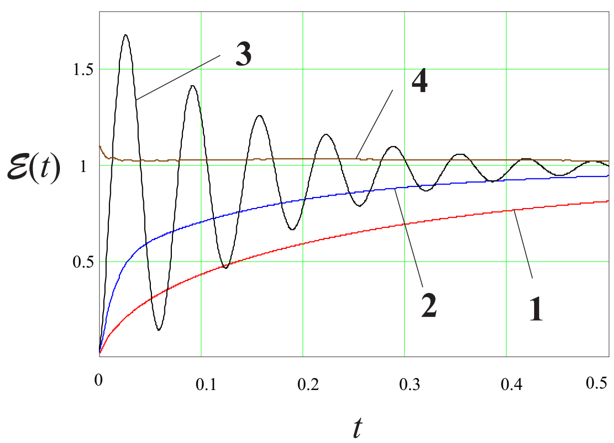

Figure 1. Solution of the creep problem for several parameter values.

Table 1. The values of parameters for example shown in Figure 1.

	k_1	k_2	k_3	β	α_1	α_2	α_3	$\tau_1 = \tau_2 = \tau_3 = k$
1	1	1	1	0.9	0.2	0.5	0.9	1
2	1	−0.8	1	0.9	0.2	0.5	0.9	1
3	1	−1.7	1	0.9	0.2	0.5	0.9	1
4	1	−3	1	0.9	0.2	0.5	0.9	1

For lines **1–3** in Figure 1, the restrictions in Equation (20) are satisfied. In case **4**, the restrictions are violated. The interesting fact is that in case **3**, although the dissipation inequality is satisfied, the creep curve is oscillatory with a decreasing amplitude. This is due to the negative value of k_2, which makes that internal variable ε_2 bring energy into the system. In addition, from the final value theorem (see [24] p. 40), we have the following final values for $\mathcal{E}(t)$:

$$\begin{aligned}\lim_{t\to\infty}\mathcal{E}(t) &= \lim_{s\to 0} s\overline{\mathcal{E}}(s) = \lim_{s\to 0}\frac{1}{k + s^{\beta}\sum\limits_{j=1}^{3}\frac{k_j}{s^{\alpha_j}+\frac{1}{\tau_j}}} = \frac{1}{k},\\ \lim_{t\to 0}\mathcal{E}(t) &= \lim_{s\to \infty} s\overline{\mathcal{E}}(s) = \lim_{s\to \infty}\frac{1}{k + s^{\beta}\sum\limits_{j=1}^{3}\frac{k_j}{s^{\alpha_j}+\frac{1}{\tau_j}}} = 0.\end{aligned} \tag{31}$$

Our numerical results are in agreement with Equation (31).

- As a final example, we present the wave equation for model B. Regarding the dimensionless form of the equation of motion, the constitutive Equations (12) and (13) and geometrical conditions for the spatially one dimensional case are

$$\frac{\partial}{\partial x}\sigma(t,x) + f(x,t) = \frac{\partial^2}{\partial t^2}u(t,x), \tag{32}$$

$$\sigma(t,x) = k\mathcal{E}(t,x) + \mu\int_0^B \varepsilon_\alpha(t,x)d\alpha, \tag{33}$$

$$\tau^{\alpha}\,{}_0D_t^{\alpha}\varepsilon_\alpha(t,x) + \varepsilon_\alpha(t,x) = \tau_1^{\beta}\,{}_0D_t^{\beta}\mathcal{E}(t,x), \quad \varepsilon_\alpha(0,x) = 0, \tag{34}$$

$$\mathcal{E}(x,t) = \frac{\partial}{\partial x}u(x,t), \qquad x\in(-\infty,\infty), \;\; t>0, \tag{35}$$

where $f(x,t)$ denotes the body force. By applying the Laplace transform to Equation (33) and (34), and by using (35), we obtain

$$\begin{aligned}\overline{\sigma}(x,s) &= k\overline{\mathcal{E}}(x,s) + \mu(\tau_1 s)^{\beta}\overline{\mathcal{E}}(x,s)\int_0^B\frac{d\alpha}{1+(\tau s)^{\alpha}}\\ &= \overline{\frac{\partial u}{\partial x}}(x,s)\left[k + \mu(\tau_1 s)^{\beta}\frac{\ln\frac{2\tau S}{1+\tau s}}{\ln \tau s}\right].\end{aligned}$$

Therefore, we get

$$\sigma(x,t) = \frac{\partial u(x,t)}{\partial x} \underset{t}{*} A(t), \tag{36}$$

where

$$A(t) = \mathcal{L}^{-1}\left(k + \mu(\tau_1 s)^{\beta}\frac{\ln\frac{2\tau s}{1+\tau s}}{\ln \tau s}\right)(t),$$

and $\underset{t}{*}$ denotes convolution with respect to t (i.e., $f \underset{t}{*} g = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$). By combining Equations (32)–(35) with Equation (36), we obtain the generalized wave equation:

$$\frac{\partial^2}{\partial t^2} u(t,x) = \frac{\partial^2 u(t,x)}{\partial x^2} \underset{t}{*} A(t), \tag{37}$$

subject to the initial

$$u(t,x) = u_0(x), \qquad \frac{\partial}{\partial t} u(t,x)\bigg|_{t=0} = v_0(x), \tag{38}$$

and boundary conditions

$$\lim_{x \to \pm\infty} u(t,x) = 0.$$

The system in Equations (37) and (38) may be written (see [25]) as

$$\frac{\partial^2}{\partial t^2} u(t,x) = \frac{\partial^2 u(t,x)}{\partial x^2} \underset{t}{*} A(t) + \delta'(t)u_0(x) + \delta(t)v_0(x), \tag{39}$$

where δ is the Dirac delta distribution. The solution to Equation (39) follows from expression 4.7 in [26], where $A(s)$ is replaced with $s^2\left[k + \mu(\tau_1 s)^\beta \frac{\ln \frac{2\tau S}{1+\tau s}}{\ln \tau s}\right]$). We obtain the solution to Equation (39), expressed as

$$u(t,x) = \mathcal{L}^{-1}\left[\frac{1}{2} s\sqrt{k + \mu(\tau_1 s)^\beta \frac{\ln \frac{2\tau s}{1+\tau s}}{\ln \tau s}} \exp\left(-|x| s\sqrt{k + \mu(\tau_1 s)^\beta \frac{\ln \frac{2\tau s}{1+\tau s}}{\ln \tau s}}\right) \underset{x}{*} \frac{s u_0(x) + v_0(x)}{s^2}\right]. \tag{40}$$

Specifically, we solved Equation (39) for the special initial conditions $u_0(x) = \delta(x)$, $v_0(x) = 0$. Then, Equation (40) leads to

$$\begin{aligned} u(x,t) \quad = \quad & \lim_{P\to\infty} \frac{1}{4\pi} \int_{-P}^{P} \exp(t(x_0+ip)) \left[\sqrt{k + \mu(\tau_1(x_0+ip))^\beta \frac{\ln \frac{2\tau(x_0+ip)}{1+\tau(x_0+ip)}}{\ln \tau(x_0+ip)}} \times \right. \\ & \left. \times \exp\left(-|x|(x_0+ip)\sqrt{k + \mu(\tau_1(x_0+ip))^\beta \frac{\ln \frac{2\tau(x_0+ip)}{1+\tau(x_0+ip)}}{\ln \tau(x_0+ip)}}\right)\right] dp. \end{aligned} \tag{41}$$

The results for the numerical inversion of Equation (41) for two specific choices of parameters and for three time instants are shown in Figure 2. This figure shows that we may define a wave speed as the speed of propagation of the maxima in Figure 2 (i.e., the propagation speed is the speed of propagation of the maximum point of the Green function) [27]. Another way to define the propagation speed is presented in [28].

In Figure 3, we used $k = 0$ in Equation (41). Then, the material described by (12) is fluid like. Again, the solution is shown for three time instants.

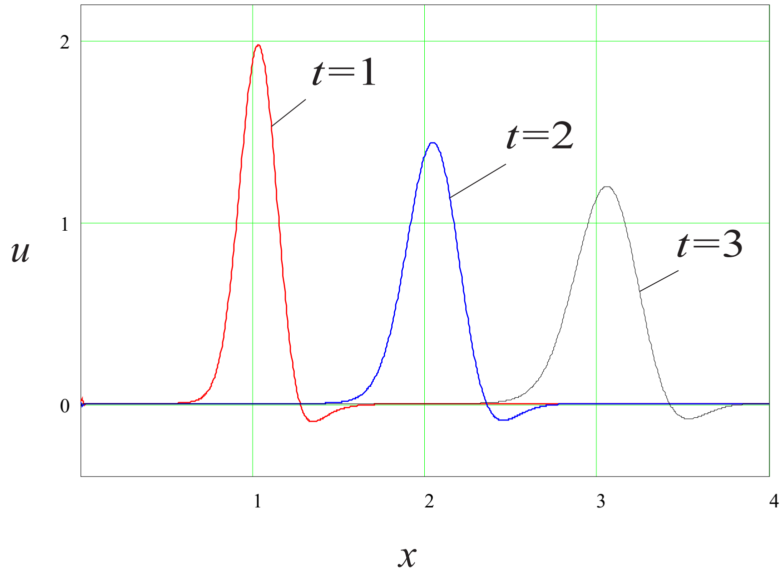

Figure 2. Solution of the wave equation (Equation (41)) for the case $k = 1$.

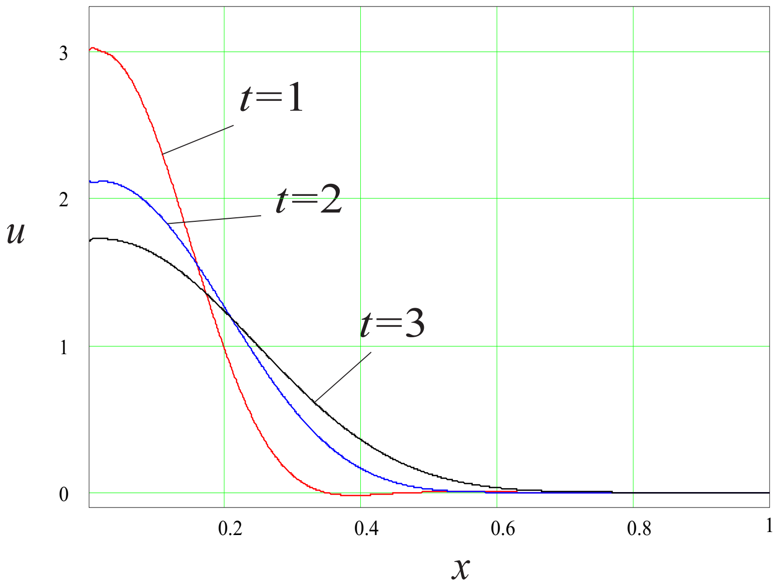

Figure 3. Solution of the wave equation (Equation (41)) for the case $k = 0$.

5. Conclusions

1. In this work, we studied a fractional generalization of the internal variable method. We proposed two generalizations of the standard constitutive equation of internal variable viscoelasticity, given by Equations (8) and (9) for one and Equations (12) and (13) for the other. The main results are the thermodynamical admissibility conditions given by Equations (20) and (23). We also presented two numerical examples illustrating the creep and wave propagation in the models proposed here.

2. We used the Rieman–Liouville fractional derivatives. For the possibility of the use of the Caputo–Fabrizio derivative, having non-singular kernels, see recent articles such as [29,30]. Additionally, for the application of *general fractional derivatives* as proposed in [13], the work in [31] for the internal variable viscoelasticity model proposed here seems to be an interesting possibility.For example, the use of Sonin kernels in constitutive equations and the formulation of the thermodynamical restrictions that follow constitutes an interesting problem that we are working on.

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