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Srednje vrednosti multiplikativnih aritmetičkih funkcija više promenljivih zavisnih od NZD i NZS argumenata

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Mean values of multiplicative arithmetic functions of several variables depending on GCD and LCM of arguments

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Abstract

In this thesis we give some new asymptotic formulas for mean values of multiplicative functions of several variables depending on GCD and LCM of arguments.

We obtain an asymptotic formula with a power saving error term for the summation function of a family of generalized least common multiple and greatest common divisor functions of several integer variables.

$$\sum_{n_1,\ldots,n_{k+\ell}\le x}\left[\frac{[n_1,\ldots,n_k]^a}{(n_1,\ldots,n_k)^c},\frac{[n_{k+1},\ldots,n_{k+\ell}]^b}{(n_{k+1},\ldots,n_{k+\ell})^d}\right]$$

$$=\frac{C_{k,a,c;\ell,b,d}}{(a+1)^k(b+1)^\ell}\quad x^{k(a+1)+\ell(b+1)}+O_\epsilon\left(x^{k(a+1)+\ell(b+1)-\frac{1}{2}+\epsilon}\right)$$

and

$$\sum_{n_1,\ldots,n_{k+\ell}\le x}\frac{\left[\frac{[n_1,\ldots,n_k]^a}{(n_1,\ldots,n_k)^c},\frac{[n_{k+1},\ldots,n_{k+\ell}]^b}{(n_{k+1},\ldots,n_{k+\ell})^d}\right]}{(n_1\ldots n_k)^a(n_{k+1}\ldots n_{k+\ell})^b}=C_{k,a,c;\ell,b,d}\;x^{k+\ell}+O_\epsilon\left(x^{k+\ell-\frac{1}{2}+\epsilon}\right).$$

Also we obtain an asymptotic formula with a power saving error term for the summation function of Euler phi-function evaluated at iterated and generalized least common multiples of four integer variables.

$$\sum_{n_1,n_2,n_3,n_4\le x}\varphi\left(\left[\frac{[n_1,n_2]^a}{(n_1,n_2)^c},\frac{[n_3,n_4]^b}{(n_3,n_4)^d}\right]\right)$$
$$=\frac{C_{a,c;b,d}}{(a+1)^2(b+1)^2}\;x^{2a+2b+4}+O_\epsilon\left(x^{2a+2b+\frac{7}{2}+\epsilon}\right).$$

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CONTENTS

1. ***Arithmetic functions*** . . . 1
1.1 **Examples of arithmetic functions** . . . 2
1.2 **The Dirichlet convolution of arithmetic functions** . . . 8
1.2.1 **Dirichlet inverse and the Möbius inversion formula** . . . 10

2. ***Summation of arithmetic functions*** . . . 12
2.1 **Partial summation** . . . 12
2.2 **Dirichlet hyperbola method** . . . 14
2.3 **The average order of the sum of divisors function** . . . 17
2.4 **The average order of Euler's φ-function** . . . 18
2.5 **Generating Dirichlet series** . . . 19
2.6 **The Riemann zeta function** . . . 22
2.7 **Multiplicative arithmetic functions of several variables** 25

3. ***On some multivariate LCM and GCD sums*** . . . 28
3.1 **Introduction** . . . 28
3.2 **Proof of Theorem 1** . . . 31
3.3 **Proofs of Corollaries** . . . 36
3.4 **An example with four variables** . . . 39

4. ***On some multivariate summatory functions of the Euler phi-function*** . . . 49
4.1 **Introduction** . . . 49
4.2 **Proof of Theorem 2** . . . 51

1. ARITHMETIC FUNCTIONS

In this chapter, we will recall some of the well known facts about arithmetic functions of one integer variable and give some representative examples. We focus mainly on multiplicative arithmetic functions. Also we will discuss the Dirichlet convolution of arithmetic functions of one variable and present some properties which are related to it.

Definition 1. An arithmetic function is a complex-valued function defined on $\mathbb{N} = \{1, 2, 3, \dots\}$

$$f : \mathbb{N} \to \mathbb{C}.$$

Two classes of arithmetic functions play particularly important roles: additive functions and multiplicative functions.

Definition 2. An arithmetic function is called *additive* if it satisfies

$$f(mn) = f(m) + f(n)$$

for all $m, n \in \mathbb{N}$ such that $\gcd(m, n) = 1$.

Definition 3. An arithmetic function is said to be *multiplicative* if $f(1) = 1$ and

$$f(mn) = f(m)f(n)$$

for all $m, n \in \mathbb{N}$ such that $\gcd(m, n) = 1$.

The null function is not multiplicative, because it does not satisfy the condition $f(1) = 1$.

A function f is called *completely additive* or *completely multiplicative* if the conditions above hold even when $(m, n) \neq 1$, that is if $f(p^v) = vf(p)$ or $f(p^v) = f(p)^v$, respectively.

And f is called *strongly additive* or *strongly multiplicative* if, in addition to the conditions above, we have $f(p^v) = f(p)$ for all $v \geq 1$.

The following proposition is useful in constructing new multiplicative functions from known ones.

Proposition 1. *Let f be a multiplicative arithmetic function and*

$$F(n) = \sum_{d|n} f(d).$$

Then F is also a multiplicative arithmetic function.

Proof. Let m and n be two coprime integers. If $d|mn$, the divisor d can be expressed uniquely as a product $d_1 \cdot d_2$ with $d_1|m$ and $d_2|n$. Conversely, if $d_1|m$ and $d_2|n$, therefore $d_1d_2|mn$. Obviously, if m and n are coprime, so their divisors are d_1 and d_2.
Then

$$\begin{aligned}
F(mn) &= \sum_{d|mn} f(d) \\
&= \sum_{d_1|m, d_2|n} f(d_1d_2) \\
&= \sum_{d_1|m, d_2|n} f(d_1)f(d_2) \\
&= \sum_{d_1|m} f(d_1) \sum_{d_2|n} f(d_2) \\
&= F(m)F(n).
\end{aligned}$$

□

1.1 Examples of arithmetic functions

Here we will define and present some examples of important arithmetic functions.

1. The counting functions

- The counting function of the total number of prime factors of an integer n:

$$\Omega(n) = \sum_{p^v \| n} v.$$

- The counting function of the number of distinct prime factors of an integer n:

$$\omega(n) = \sum_{p^v \| n} 1 = \sum_{p|n} 1.$$

It is immediate from their definitions that Ω and ω are additive, the former completely, the latter strongly.

2. The sum of the k-th powers of the positive divisors of n including 1 and n

$$\sigma_k(n) = \sum_{d|n} d^k \quad (k \in \mathbb{C}).$$

- The sum of divisors function.

$$\sigma(n) = \sigma_1(n) = \sum_{d|n} d \quad (k = 1).$$

- The number of divisors function.

$$\tau(n) = \sigma_0(n) = \sum_{d|n} 1 \quad (k = 0).$$

Proposition 2. *The function σ_k is multiplicative.*

Proof. Since we have $\gcd(m, n) = 1$, then every divisor d of m, n can be written uniquely as $d = d_1 d_2$ such that $d_1 | m$ and $d_2 | n$.

Therefore

$$\begin{aligned}\sigma_k(mn) &= \sum_{d|mn} d^k \\ &= \sum_{d_1|m, d_2|n} (d_1 d_2)^k \\ &= \left(\sum_{d_1|m} d_1^k\right)\left(\sum_{d_2|n} d_2^k\right) \\ &= \sigma_k(m)\sigma_k(n).\end{aligned}$$

□

3. Euler's totient function, counts the number of invertible residues modulo n, that is:

$$\varphi(n) = \sum_{1 \le h \le n, (h,n)=1} 1.$$

Proposition 3. *The Euler's totient function is a multiplicative, i.e., if gcd$(m, n) = 1$ then $\varphi(mn) = \varphi(m)\varphi(n)$.*

- If p is prime and $k > 0$

$$\varphi(p^k) = p^k - p^{k-1}$$

  and when $k = 1$, then we have

$$\varphi(p) = p - 1.$$

- The sum of the Euler's totient function over all positive divisors of n (including 1 and n itself) is n.

  Proposition 4. *For any positive integer n we have:*

$$\sum_{d|n} \varphi(d) = n.$$

Proof. Let

$$F(n) = \sum_{d|n} \varphi(d).$$

As φ is multiplicative function, so is $F(n)$ by Proposition 1.
For integers which are prime power, i.e, of the form p^k for some $k \geq 1$,

$$\begin{aligned} F(p^k) &= \varphi(1) + \varphi(p) + \varphi(p^2) + \cdots + \varphi(p^k) \\ &= 1 + (p-1) + (p^2 - p) + \cdots + p^k - p^{k-1} \\ &= p^k. \end{aligned}$$

Now consider any integer n, and consider its prime factorization. Then,

$$n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}, \quad k_i \geq 1$$

$$\begin{aligned} \implies F(n) &= F(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}) \\ &= F(p_1^{k_1}) F(p_2^{k_2}) \cdots F(p_r^{k_r}) \\ &= p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} \\ &= n. \end{aligned}$$

□

- Because of multiplicativity of Euler's function we have the following explicit formula.

Proposition 5. *For any positive integer n, we have:*

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

4. Möbius function

The Möbius function is defined by:

$$\mu(n) = \begin{cases} (-1)^{\omega(n)} & if \quad n \quad is \quad squarefree, \\ 0 & otherwise. \end{cases}$$

For example, the values of this function at the first few positive integers are:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\mu(n)$	1	-1	-1	0	-1	1	-1	0	0	1	-1	0	-1	1	1

n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$\mu(n)$	0	-1	0	-1	0	1	1	-1	0	0	1	0	0	-1	-1

Proposition 6. *The Möbius function is multiplicative, that is,* $\mu(mn) = \mu(m)\mu(n)$*, whenever* $(m, n) = 1$.

Proof. Let m and n be two coprime integers. We consider the following two cases.
Case 1: Suppose $\mu(mn) = 0$. Therefore there is a prime number p such that $p^2|mn$. Since m and n are coprime, so p can not divide both integers m and n. Hence either $p^2|m$ or $p^2|n$. Then, either $\mu(m) = 0$ or $\mu(n) = 0$, and we conclude that $\mu(mn) = \mu(m)\mu(n)$.

Case 2: Let $\mu(mn) \neq 0$. Therefore, mn is squarefree, hence so are m and n.
We can write m, n as $m = p_1p_2 \dots p_r$ and $n = q_1q_2 \dots q_s$ where p_i and q_j are all distinct primes. Then, $mn = p_1p_2 \dots p_rq_1q_2 \dots q_s$ where all the primes occurring in the factorization of mn are distinct. Hence,

$$\mu(mn) = (-1)^{r+s} = (-1)^r(-1)^s = \mu(m)\mu(n).$$

□

- The sum of the Möbius function over all positive divisors of n (including 1 and n itself) is zero except when $n = 1$.

Proposition 7. *For any positive integer n we have:*

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & if \quad n = 1, \\ 0 & if \quad n > 1. \end{cases}$$

Proof. Let

$$F(n) = \sum_{d|n} \mu(d).$$

Since μ is multiplicative, so is $F(n)$ by Proposition 1.
Obviously,

$$F(1) = \sum_{d|1} \mu(d) = \mu(1) = 1.$$

Now consider the integers which are prime powers, i.e., of the form p^k for some $k \geq 1$,

$$\begin{aligned} F(p^k) &= \mu(1) + \mu(p) + \mu(p^2) + \cdots + \mu(p^k) \\ &= 1 + (-1) + 0 + \cdots + 0 \\ &= 0. \end{aligned}$$

And for any integer n, consider its prime factorization.
Therefore

$$n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}, \quad k_i \geq 1$$

$$\begin{aligned} \implies F(n) &= \prod_i F(p_i^{k_i}) \\ &= 0. \end{aligned}$$

□

5. The von Mangoldt function
   The von Mangoldt function is defined by:

$$\Lambda(n) = \begin{cases} \ln p & if \quad n = p^v \quad for \quad some \quad v \geq 1, \\ 0 & otherwise. \end{cases}$$

   - For example, the values of this function at the first few positive integers are:

n	1	2	3	4	5	6	7	8	9	10	11
$\Lambda(n)$	0	$\log 2$	$\log 3$	$\log 2$	$\log 5$	0	$\log 7$	$\log 2$	$\log 3$	0	$\log 11$

n	12	13	14	15	16	17	18	19	20	21	22
$\Lambda(n)$	0	$\log 13$	0	0	$\log 2$	$\log 17$	0	$\log 19$	0	0	0

n	23	24	25	26	27	28	29	30
$\Lambda(n)$	$\log 23$	0	$\log 5$	0	$\log 3$	0	$\log 29$	0

   - The sum of the von Mangoldt function over all divisors of n (including 1 and n itself) is $\log n$.

Proposition 8. *For any positive integer n we have:*

$$\sum_{d|n} \Lambda(d) = \log n.$$

Remark 1. *The von Mangoldt function is neither multiplicative nor additive.*

1.2 *The Dirichlet convolution of arithmetic functions*

Definition 4. If f and g are two arithmetic functions, the Dirichlet convolution of f and g, denoted by $f * g$, is the arithmetic function defined by

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d) = \sum_{d|n} f(n/d)g(d).$$

Remark 2. *We can write the Dirichlet convolution also as follows*

$$(f * g)(n) = \sum_{mk=n} f(m)g(k),$$

where m and k run over all pairs of positive integers whose product equals n.

Proposition 9. *Let f and g are two multiplicative arithmetic functions. Then, $f * g$ is also multiplicative.*

Proof. As we did in Proposition 1, for coprime integers m and n,

$$\begin{aligned}
(f * g)(mn) &= \sum_{d|mn} f(d)g\left(\frac{mn}{d}\right) \\
&= \sum_{d_1d_2|mn} f(d_1d_2)g\left(\frac{mn}{d_1d_2}\right) \\
&= \sum_{d_1|m, d_2|n} f(d_1)f(d_2)g\left(\frac{m}{d_1}\right)g\left(\frac{n}{d_2}\right) \\
&= \sum_{d_1|m}\sum_{d_2|n} f(d_1)f(d_2)g\left(\frac{m}{d_1}\right)g\left(\frac{n}{d_2}\right) \\
&= \left(\sum_{d_1|m} f(d_1)g\left(\frac{m}{d_1}\right)\right)\left(\sum_{d_2|n} f(d_2)g\left(\frac{n}{d_2}\right)\right)
\end{aligned}$$

which implies the Proposition.

□

Example 1. In the following, we offer some examples of arithmetic functions which can be expressed as Dirichlet convolutions.

1. $\tau(n) = \sum_{d|n} 1, \tau = 1 * 1$, where 1 is the arithmetic function defined by $1(n) = 1$.

2. $\sigma(n) = \sum_{d|n} d, \sigma = 1 * id$, where id is the identity function such that $id(n) = n$.

3. $\sum_{d|n} \mu(d) = \delta(n), \mu * 1 = \delta$, where δ is the unit function such that $\delta(n) = 1$ if $n = 1$ and $\delta(n) = 0$ if $n > 1$.

4. $\sum_{d|n} \mu(d)(n/d) = \varphi(n), \mu * id = \varphi$.

5. $\sum_{d|n} \varphi(d) = n, \varphi * 1 = id$.

6. $\sum_{d|n} \Lambda(d) = \ln n, \Lambda * 1 = \log$.

1.2.1 ***Dirichlet inverse and the Möbius inversion formula***

Proposition 10. *[4, Th. 2.8] Let f be an arithmetic function with $f(1) \neq 0$, then f has a unique arithmetical function f^{-1} called the Dirichlet inverse of f, such that*

$$f * f^{-1} = f^{-1} * f = \delta.$$

Furthermore, f^{-1} is given by the recursion formulas

$$f^{-1}(1) = \frac{1}{f(1)},$$

$$f^{-1}(n) = \frac{-1}{f(1)} \sum_{\substack{d|n \\ d<n}} f\left(\frac{n}{d}\right) f^{-1}(d) \quad for \quad n > 1.$$

Example 2. We have $\mu * 1 = \delta$, therefore the Möbius function is the Dirichlet inverse of the function 1.

Remark 3. *We have $(f * g)(1) = f(1)g(1)$. Therefore, if $f(1) \neq 0$ then $(f * g)(1) \neq 0$. So this fact inform us that, the set of all arithmetical functions f with $f(1) \neq 0$ forms an abelian group with respect to the operation $*$, and the unity being the function δ.*

Proposition 11. *(Möbius inversion formula). If we have*

$$f(n) = \sum_{d|n} g(d) \tag{1.1}$$

This equation implies

$$g(n) = \sum_{d|n} f(d)\mu\left(\frac{n}{d}\right). \tag{1.2}$$

Conversely, (1.2) implies (1.1).

Proof. Equation (1.1) can be written as $f = g*1$. Multiplying each side of the equation by the function μ we obtain $f*\mu = (g*1)*\mu = g*(1*\mu) = g*\delta = g$, which is (1.2). Conversely, multiplying $f * \mu = g$ by 1 gives (1.1). □

2. SUMMATION OF ARITHMETIC FUNCTIONS

In number theory, a summation function $F(x)$ of an arithmetic function $f(n)$ is defined by

$$F(x) = \sum_{n \leq x} f(n).$$

Definition 5. Let $f(n)$ be an arithmetic function. It is said that $f(n)$ has a mean value (average value) c if the following limit

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n)$$

exists and is equal to c.

In this chapter we will review some of the well known summation tools which are used to evaluate sums of arithmetic functions. Also we give examples of the summations of some arithmetic functions and estimate their average order.

2.1 ***Partial summation***

In this section we derive an identity used in many situation. Partial summation is an extremely useful and has various applications in number theory and analysis. The partial summation identity yields a formula that is very convenient for estimating sums of arithmetic functions.

Proposition 12. *(Partial summation formula, see [28]) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real or complex numbers, and $f(t)$ differentiable function for $t \geq 0$, and set $A(x) = \sum_{n \leq x} a_n$. Then*

$$\sum_{n \leq x} a_n f(n) = A(x) f(x) - \int_1^x A(t) f'(t) dt.$$

Now we will estimate the function $\sum_{n\le x}\frac{1}{n}$ by applying the partial summation.

Example 3. Let $x \geq 1$, then

$$\sum_{n\le x}\frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$$

where γ is Euler's constant.

We will apply Partial summation formula by taking $a_n = 1$ for $n \in \mathbb{N}, A(x) = [x]$ and $f(t) = \frac{1}{t}$, then

$$\sum_{n\le x}\frac{1}{n} = [x]\frac{1}{x} + \int_1^x \frac{[t]dt}{t^2} \tag{2.1}$$

So, we can write the function $[x]$ as $[x] = x - \{x\}$ in (2.1), where $\{x\}$ denotes the fractional part of x.
Therefore

$$\begin{aligned}\sum_{n\le x}\frac{1}{n} &= \frac{x-\{x\}}{x} + \int_1^x \frac{t-\{t\}dt}{t^2}\\ &= 1 + O\left(\frac{1}{x}\right) + \log x - \int_1^x \frac{\{t\}dt}{t^2}\\ &= \log x + 1 - \left(\int_1^\infty \frac{\{t\}dt}{t^2} - \int_x^\infty \frac{\{t\}dt}{t^2}\right) + O\left(\frac{1}{x}\right)\\ &= \log x + 1 - \int_1^\infty \frac{\{t\}dt}{t^2} + O\left(\frac{1}{x}\right).\end{aligned}$$

Where

$$\int_x^\infty \frac{\{t\}dt}{t^2} \ll \int_x^\infty \frac{dt}{t^2} = \frac{1}{x}.$$

This is precisely the formula we need, if we denote $\gamma = 1 - \int_1^\infty \frac{\{t\}dt}{t^2}$.

Letting $x \to \infty$ in Example 3, we find that

$$\lim_{x\to\infty}\left(\sum_{n\le x}\frac{1}{n} - \log x\right) = 1 - \int_1^{\infty}\frac{\{t\}dt}{t^2},$$

and the value of γ is equal 0.5772156649 ... .

2.2 Dirichlet hyperbola method

Here we will explore the average order of the divisor function $\tau(n)$. A simple interchange of summations gives a first estimate. Indeed we have

$$\begin{aligned}\sum_{n\le x}\tau(n) &= \sum_{n\le x}\sum_{d|n} 1\\ &= \sum_{d\le x}\sum_{qd\le x} 1\\ &= \sum_{d\le x}\left[\frac{x}{d}\right]\\ &= \sum_{d\le x}\left(\frac{x}{d} + O(1)\right)\\ &= x\sum_{\le x}\frac{1}{d} + O(x)\\ &= x\log x + O(x).\end{aligned}$$

We can improve this estimate by using so called "Dirichlet hyperbola method", invented by Dirichlet to evaluate the sums of the arithmetic functions.

The Dirichlet hyperbola method is a technique in number theory to estimate the sum $\sum_{n\le x} f(n)$, where f is an arithmetic function of the form $f = g * h$. That is

$$f(n) = \sum_{d|n} g(d)h(n/d)$$

for two arithmetic functions g and h. We denote by

$$G(x) = \sum_{n\le x} g(n),$$

$$H(x) = \sum_{n \le x} h(n),$$

the corresponding summation functions.

Proposition 13. *If f is an arithmetic function and g, h are two arithmetic functions with summatory functions G, H respectively, then for any parameter y, $1 \le y \le x$, we have*

$$\sum_{n \le x} f(n) = \sum_{a \le y} g(a) H\left(\frac{x}{a}\right) + \sum_{b \le \frac{x}{y}} h(b) G\left(\frac{x}{b}\right) - G(y) H\left(\frac{x}{y}\right).$$

Proof. We have

$$\begin{aligned}
\sum_{n \le x} f(n) &= \sum_{ab \le x} g(a)h(b) \\
&= \sum_{\substack{ab \le x \\ a \le y}} g(a)h(b) + \sum_{\substack{ab \le x \\ a > y}} g(a)h(b) \\
&= \sum_{a \le y} g(a) H\left(\frac{x}{a}\right) + \sum_{b \le \frac{x}{y}} h(b) \left\{G\left(\frac{x}{b}\right) - G(y)\right\} \\
&= \sum_{a \le y} g(a) H\left(\frac{x}{a}\right) + \sum_{b \le \frac{x}{y}} h(b) G\left(\frac{x}{b}\right) - G(y) H\left(\frac{x}{y}\right).
\end{aligned}$$

□

We will evaluate the partial sums of the number of divisors function by applying the Dirichlet hyperbola method which give us more precise information on the average behavior of $\tau(n)$.

Example 4. Let $x \ge 1$, then

$$\sum_{n \le x} \tau(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x})$$

where $\tau(n)$ is the number of divisors function and γ is Euler's constant.

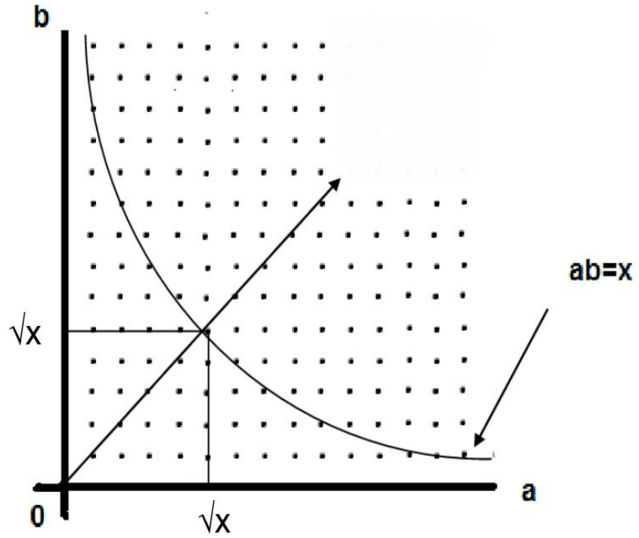

Fig. 2.1: The equilateral hyperbola and the lattice of integral points.

First we have

$$\sum_{n\leq x}\tau(n)=\sum_{n\leq x}\sum_{ab=n}1=\sum_{\substack{a,b\leq x\\ ab\leq x}}1.$$

Now, we can apply the hyperbola method where $g(n)=1, h(n)=1$ and $G(x)=[x], H(x)=[x]$ as follows:

$$\sum_{n\le x}\tau(n) = \sum_{a\le y} g(a)H\left(\frac{x}{a}\right) + \sum_{b\le \frac{x}{y}} h(b)G\left(\frac{x}{b}\right) - G(y)H\left(\frac{x}{y}\right)$$
$$= \sum_{a\le y}\left[\frac{x}{a}\right] + \sum_{b\le \frac{x}{y}}\left[\frac{x}{b}\right] - [y]\left[\frac{x}{y}\right]$$
$$= x\sum_{a\le y}\frac{1}{a} + O(y) + x\sum_{b\le \frac{x}{y}}\frac{1}{b} + O\left(\frac{x}{y}\right) - (y+O(1))\left(\frac{x}{y}+O(1)\right).$$

So

$$\sum_{n\le x}\tau(n) = x(\log\sqrt{x}+\gamma+O\left(\frac{1}{\sqrt{x}}\right)) + x(\log\sqrt{x}+\gamma+O\left(\frac{1}{\sqrt{x}}\right)) - x + O(\sqrt{x})$$
$$= x\log x + (2\gamma-1)x + O(\sqrt{x}).$$

Which is the desired estimate.

In the following we will give more examples of the summations of arithmetic functions and estimate the average order of that functions.

2.3 ***The average order of the sum of divisors function***

The following example evaluates the average order of the function

$$\sigma(n) = \sum_{d|n} d.$$

Example 5. Let $x \ge 1$, then we have

$$\sum_{n\le x}\sigma(n) = \frac{1}{12}\pi^2x^2 + O(x\log x).$$

The idea of solving this example is to write the function under consideration as a sum over the divisors of n and then interchange summations. Thus we have

$$\begin{aligned}\sum_{n\leq x}\sigma(n) &= \sum_{n\leq x}\sum_{d|n} d\\ &= \sum_{d\leq x}\sum_{qd\leq x} q\\ &= \frac{1}{2}\sum_{d\leq x}\left[\frac{x}{d}\right]\left(\left[\frac{x}{d}\right]+1\right)\\ &= \frac{1}{2}\sum_{d\leq x}\frac{x^2}{d^2} + O\left(x\sum_{d\leq x}\frac{1}{d}\right).\end{aligned}$$

It gives the following formula

$$\sum_{d\geq 1}\frac{1}{d^2} = \frac{\pi^2}{6}. \tag{2.2}$$

This implies the stated formula.

2.4 *The average order of Euler's φ-function*

In the following example we will estimate the average order of the function $\varphi(n)$.

Example 6. Let $x > 1$, then we have

$$\sum_{n\leq x}\varphi(n) = \frac{3}{\pi^2}x^2 + O(x\log x).$$

We start by using the formula

$$\varphi(n) = n\sum_{d|n}\frac{\mu(d)}{d}.$$

Then

$$\begin{aligned}
\sum_{n\le x}\varphi(n) &= \sum_{n\le x}\sum_{d|n}\mu(d)n/d \\
&= \sum_{d\le x}\mu(d)\sum_{qd\le x} q \\
&= \sum_{d\le x}\mu(d).\frac{1}{2}\left[\frac{x}{d}\right]\left(\left[\frac{x}{d}\right]+1\right) \\
&= \sum_{d\le x}\mu(d)\left(\frac{1}{2}\left(\frac{x}{d}\right)^2+O\left(\frac{x}{d}\right)\right) \\
&= \frac{1}{2}x^2\sum_{d\le x}\frac{\mu(d)}{d^2}+O\left(x\sum_{d\le x}\frac{1}{d}\right).
\end{aligned}$$

We know that the Möbius function is the inverse of 1 under convolution, so we have

$$\left(\sum_{d\ge 1}\frac{\mu(d)}{d^2}\right)\left(\sum_{d\ge 1}\frac{1}{d^2}\right)=1.$$

Where the two series are absolutely convergent, and by (2.2) in the Example 5 we have

$$\sum_{d\ge 1}\frac{\mu(d)}{d^2}=\frac{6}{\pi^2}.$$

This giving the required conclusion.

2.5 ***Generating Dirichlet series***

A Dirichlet series is an expression of the form

$$\sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

where s is a complex variable and a_n is a sequence of complex numbers.

Dirichlet series play an important role in analytic number theory and especially in the theory of arithmetic functions. If we write the a_n as the values of an arithmetic function $f(n) = a_n$, then we say that the above series is the Dirichlet series associated with f.

Definition 6. Let f be an arithmetic function. The Dirichlet series associated to f is the series

$$D(f; s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

For two Dirichlet series $D(f; s)$ and $D(g; s)$ represented respectively as follows:

$$D(f; s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad , \Re(s) > a$$

and

$$D(g; s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s} \quad , \Re(s) > b$$

where both Dirichlet series converge absolutely in the half-plane, we define the product of them in the following way by:

$$D(f; s)D(g; s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s} \tag{2.3}$$

and

$$h(n) = \sum_{mk=n} f(m)g(k) = (f * g)(n)$$

where $h = f * g$ is the Dirichlet convolution of f and g.

Because of absolute convergence of these series we can multiply them together, hence the product definition agrees with the formal calculation

$$\sum_{m=1}^{\infty} \frac{f(m)}{m^s} \sum_{k=1}^{\infty} \frac{g(k)}{k^s} = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{f(m)g(k)}{(mk)^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{mk=n} f(m)g(k).$$

Example 7. If $f(n) = 1$ for $n \in \mathbb{N}$, then

$$D(1; s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \quad , \Re(s) > 1$$

defines the Riemann zeta function.

Example 8. Taking $f(n) = 1$ and $g(n) = \mu(n)$, Möbius function, then in (2.3) we have $h(n) = \delta(n)$, so

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\delta(n)}{n^s} = 1$$

therefore

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} \quad , \Re(s) > 1.$$

Example 9. Taking $f(n) = 1$ and $g(n) = \varphi(n)$, Euler's totient function, then $h(n) = n$, so

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{n}{n^s} = \zeta(s-1)$$

therefore

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} \quad , \Re(s) > 2.$$

Proposition 14. *Let $D(f; s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ converges absolutely for $\Re(s) > a$. If f is a multiplicative function, then the Euler product is:*

$$D(f; s) = \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \dots\right).$$

Proof. For each integer n, there is a unique expression in the form

$$n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}.$$

Therefore, we have

$$\begin{aligned} D(f;s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} &= \sum_{n=1}^{\infty} \frac{f(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r})}{(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r})^s} \\ &= \sum_{n=1}^{\infty} \frac{f(p_1^{k_1}) f(p_2^{k_2}) \cdots f(p_r^{k_r})}{p_1^{sk_1} p_2^{sk_2} \cdots p_r^{sk_r}} \\ &= \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \ldots\right). \end{aligned}$$

□

If f is a completely multiplicative function, then each factor in the product is a geometric series. Therefore the Euler product becomes as following:

$$D(f;s) = \prod_p \left(1 - \frac{f(p)}{p^s}\right)^{-1}.$$

2.6 *The Riemann zeta function*

The Riemann zeta function is an important function in number theory. It is very useful for investigating properties of prime numbers especially and helpful in the proof of the prime number theorem.

Definition 7. The Riemann zeta function is the function written as $\zeta(s)$, defined in the half-plane with $\Re(s) > 1$ by the absolutely convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad , \Re(s) > 1$$

where s is a complex variable of the form $s = \sigma + it$.

As we mentioned in the last section about the Euler's product when f is a completely multiplicative function, so the Euler product of the Riemann zeta function is as follows:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \quad , \Re(s) > 1. \tag{2.4}$$

The Euler product is absolutely convergent for $\Re(s) > 1$.

Example 10. We obtain the following Euler products which are related to the Riemann zeta function and Dirichlet series by taking $f(n) = 1, \mu(n), \varphi(n), \lambda(n)$, respectively

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}} \quad, \Re(s) > 1.$$

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p (1-p^{-s}) \quad, \Re(s) > 1.$$

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \prod_p \frac{1-p^{-s}}{1-p^{1-s}} \quad, \Re(s) > 2.$$

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p \frac{1}{1+p^{-s}} \quad, \Re(s) > 1.$$

Where $\lambda(n)$ is the Liouville lambda function which is defined as $\lambda(n) = (-1)^{\Omega(n)}$.

Now taking the logarithms of equation (2.4) and using the power series $-\log(1-x) = \sum \frac{x^m}{m}$, we find that

$$\log \zeta(s) = -\sum_p \log(1-p^{-s})$$
$$= \sum_p \sum_{m=1}^{\infty} \frac{p^{-ms}}{m}.$$

By differentiating, we obtain

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \sum_{m=1}^{\infty} \frac{\log p}{p^{ms}}$$

for $\Re(s) > 1$. This is a Dirichlet series which is written as

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and its coefficient is the von Mangoldt function; therefore

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \quad , \Re(s) > 1.$$

This is fundamental relationship between prime numbers and Riemann zeta function.

We now present that the Riemann zeta function can be continued analytically beyond the line $\Re(s) = 1$ to a function which is analytic for s except for a simple pole at $s = 1$.
So we can get analytic continuation of the Riemann zeta function by applying the partial summation and taking $a_n = 1$, $f(n) = \frac{1}{n^s}$ as follows:

For $\Re(s) > 1$ we have

$$\sum_{n \le x} \frac{1}{n^s} = \frac{[x]}{x^s} + s \int_1^x \frac{[t]dt}{t^{s+1}}.$$

Let $x \to \infty$. We get

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = s \int_1^{\infty} \frac{[t]dt}{t^{s+1}}.$$

Now we can write $[t] = t - \{t\}$

$$\begin{aligned}\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} &= s \int_1^{\infty} \frac{(t - \{t\})dt}{t^{s+1}} \\ &= s \int_1^{\infty} \frac{dt}{t^s} - s \int_1^{\infty} \frac{\{t\}dt}{t^{s+1}} \\ &= \frac{s}{s-1} - s \int_1^{\infty} \frac{\{t\}dt}{t^{s+1}}.\end{aligned}$$

We noted that the integral is absolutely convergent for $\Re(s) > 0$, which is defines analytic function in this region.
Therefore the RHS of the last equation is an analytic function for $\Re(s) > 0$ except $s = 1$ where it has a simple pole.

2.7 *Multiplicative arithmetic functions of several variables*

In this section we recall some well known properties of multiplicative functions of several variables, which are needed in later chapters.

We say that an arithmetic function $f : \mathbb{N}^k \to \mathbb{C}$ in k variables ($k \geq 1$) is *multiplicative* if it is not identically zero, and we have that

$$f(m_1 n_1, \ldots, m_k n_k) = f(m_1, \ldots, m_k) f(n_1, \ldots, n_k) \tag{2.5}$$

for any $2k$ integers $m_1, \ldots, m_k, n_1, \ldots, n_k$ such that $\gcd(m_1 \ldots m_k, n_1 \ldots n_k) = 1$.
In this thesis we denote with $\mathcal{M}_k$ the set of all multiplicative functions in k variables.
Further, a function f is *completely multiplicative* if $f(1, 1, \ldots, 1) \neq 0$ and (2.5) is satisfied for any two k-tuples of integers, without restriction.

Remark 4. *In this thesis we will denote with* $(n_1, \ldots, n_k)$ *and* $[n_1, \ldots, n_k]$ *the greatest common divisor and the least common multiple of* k *integers* $n_1, \ldots, n_k$, *respectively.*

Example 11.

- The gcd function $(n_1, \ldots, n_k) \mapsto (n_1, \ldots, n_k)$ and the lcm function $(n_1, \ldots, n_k) \mapsto [n_1, \ldots, n_k]$ are multiplicative for any $k \geq 1$.
- The function $(n_1, \ldots, n_k) \mapsto \tau(n_1) \cdots \tau(n_k)$ is also multiplicative in k variables.
- The function $(n_1, \ldots, n_k) \mapsto n_1 \cdots n_k$ is completely multiplicative.

The set of all arithmetic functions $\mathcal{A}_k$ of k variables, for any $k \in \mathbb{N}$ is a complex vector space with the addition and scalar multiplication defined as usual. One can define the Dirichlet convolution in the setting of k variables by

$$(f * g)(n_1, \ldots, n_k) = \sum_{d_1 | n_1, \ldots, d_k | n_k} f(d_1, \ldots, d_k) g(n_1/d_1, \ldots, n_k/d_k).$$

Then $(\mathcal{A}_k, +, *)$ a commutative $\mathbb{C}-$algebra with multiplicative identity. The Dirichlet convolution preserves the multiplicativity of functions as in the one variable case in the first chapter.

To any arithmetic function f in k integer variables, one can associate the corresponding Dirichlet series

$$D(f; z_1, \ldots, z_k) = \sum_{n_1,\ldots,n_k=1}^{\infty} \frac{f(n_1, \ldots, n_k)}{n_1^{z_1} \cdots n_k^{z_k}},$$

which is a function of k complex variables $z_1, z_2, \ldots, z_k$. In this thesis we will use the following basic and well known, but important property: if $D(f; z_1, \ldots, z_k)$ is absolutely convergent at the point $(s_1, \ldots, s_k) \in \mathbb{C}^k$, then it is absolutely convergent for every $(z_1, \ldots, z_k) \in \mathbb{C}^k$ which satisfy

$$\Re z_j \geq \Re s_j \qquad (1 \leq j \leq k).$$

Moreover, if we have any two arithmetic functions $f, g \in \mathcal{A}_k$, such that $D(f; z_1, \ldots, z_k)$ and $D(g; z_1, \ldots, z_k)$ are absolutely convergent at some $(z_1, \ldots, z_k) \in \mathbb{C}^k$, then the Dirichlet series $D(f*g; z_1, \ldots, z_k)$ corresponding to Dirichlet convolution $f * g$ is also absolutely convergent at the same point and

$$D(f * g; z_1, \ldots, z_k) = D(f; z_1, \ldots, z_k)D(g; z_1, \ldots, z_k).$$

For multiplicative functions of several variables we have the following basic proposition, which can be found, for example at ([29]).

Proposition 15. *Let $f \in \mathcal{M}_k$. For every $(z_1, \ldots, z_k) \in \mathbb{C}^k$ the series $D(f; z_1, \ldots, z_k)$ is absolutely convergent if and only if*

$$\prod_p \sum_{\substack{v_1,\ldots,v_k=0 \\ v_1+\cdots+v_k \geq 1}}^{\infty} \frac{|f(p^{v_1}, \ldots, p^{v_k})|}{p^{v_1 \Re z_1 + \cdots + v_k \Re z_k}} < \infty.$$

In that case we have the following Euler product expansion:

$$D(f; z_1, \ldots, z_k) = \prod_p \sum_{v_1,\ldots,v_k=0}^{\infty} \frac{f(p^{v_1}, \ldots, p^{v_k})}{p^{v_1 z_1 + \cdots + v_k z_k}}. \tag{2.6}$$

Example 12. For every arithmetic function g of one integer variable, we have the following identity

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \frac{g((n_1,\ldots,n_k))}{n_1^{z_1}\cdots n_k^{z_k}} = \frac{\zeta(z_1)\cdots\zeta(z_k)}{\zeta(z_1+\cdots+z_k)} \sum_{n=1}^{\infty} \frac{g(n)}{n^{z_1+\cdots+z_k}}. \tag{2.7}$$

If we take $g(n) = n$, then (2.7) gives for $k \geq 2$ and $\Re z_1 > 1, \ldots, \Re z_k > 1$,

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \frac{(n_1,\ldots,n_k)}{n_1^{z_1}\cdots n_k^{z_k}} = \frac{\zeta(z_1)\cdots\zeta(z_k)\zeta(z_1+\cdots+z_k-1)}{\zeta(z_1+\cdots+z_k)}.$$

On the other hand, if we take $g = \delta$, we obtain in the same region

$$\sum_{\substack{n_1,\ldots,n_k=1\\(n_1,\ldots,n_k)=1}}^{\infty} \frac{1}{n_1^{z_1}\cdots n_k^{z_k}} = \frac{\zeta(z_1)\cdots\zeta(z_k)}{\zeta(z_1+\cdots+z_k)}.$$

The mean value of $f \in \mathcal{A}_k$ is defined by

$$M(f) = \lim_{x_1,\ldots,x_k\to\infty} \frac{1}{x_1\cdots x_k} \sum_{n_1\leq x_1,\ldots,n_k\leq x_k} f(n_1,\ldots,n_k),$$

if this limit exists. For example, the following theorem is proved for multiplicative functions of several variables:

Proposition 16. *(N. Ushiroya, [31, Th. 4]) Let $f \in \mathcal{M}_k$ $(k \geq 1)$ such that*

$$\sum_p \sum_{\substack{v_1,\ldots,v_k=0\\v_1+\cdots+v_k\geq 1}}^{\infty} \frac{|(\mu_k * f)(p^{v_1},\ldots,p^{v_k})|}{p^{v_1+\cdots+v_k}} < \infty.$$

Then the mean value $M(f)$ exists and

$$M(f) = \prod_p \left(1-\frac{1}{p}\right)^k \sum_{v_1,\ldots,v_k=0}^{\infty} \frac{f(p^{v_1},\ldots,p^{v_k})}{p^{v_1+\cdots+v_k}}.$$

Example 13.
- For any multiplicative function $g \in \mathcal{M}_1$, the function $(n_1,\ldots,n_k) \mapsto g((n_1,\ldots,n_k))$ has the mean value

$$M(f) = \frac{1}{\zeta(k)} \sum_{n=1}^{\infty} \frac{g(n)}{n^k}.$$

- In particular, for any $k \geq 3$, the function $(n_1,\ldots,n_k) \mapsto \varphi((n_1,\ldots,n_k))$ has the mean value $\zeta(k-1)/\zeta^2(k)$.

3. ON SOME MULTIVARIATE LCM AND GCD SUMS

In this chapter we obtain an asymptotic formula with a power saving error term for the summation function of a family of generalized least common multiple and greatest common divisor functions of several integer variables.

3.1 ***Introduction***

Let $[n_1, \ldots, n_k]$ denote the least common multiple (lcm) and $(n_1, \ldots, n_k)$ denote the greatest common divisor (gcd) of positive integers $n_1, \ldots, n_k$. Although looking simple, their statistical behavior is non-trivial. See for example the recent study [7] of the least common multiple function from the probabilistic point of view. A related and natural question would be to study asymptotic formulas for mean values of the gcd and lcm functions of several integer variables. For example, P. Diaconis and P. Erdős in [9] obtained the following asymptotic formulas in case of $k = 2$ variables:

$$\sum_{m,n \le x} (m, n) = \frac{x^2}{\zeta(2)} \left(\log x + 2\gamma - \frac{1}{2} - \frac{\zeta(2)}{2} - \frac{\zeta'(2)}{\zeta(2)} \right) + O(x^{3/2} \log x)$$

and

$$\sum_{m,n \le x} [m, n]^a = \frac{\zeta(a+2)}{(a+1)^2 \zeta(2)} x^{2a+2} + O(x^{2a+1} \log x)$$

where a is any positive real number, γ is Euler's constant and $\zeta(s)$ is the Riemann zeta function. In [16] the authors considered also the problem of establishing an asymptotic formula for the summation function of the quotient $\frac{[m,n]}{(m,n)}$ of the least common multiple and the greatest common divisor of integers m and n and obtained the formula

$$\sum_{m,n \le x} \frac{[m, n]}{(m, n)} = \frac{\pi^2}{60} x^4 + O(x^3 \log x).$$

For some interesting properties of the arithmetic function $\frac{[m,n]}{(m,n)}$ we refer the reader to the recent paper [14] and the more extensive bibliography of the related results in this area is presented in the introductory section of [16].
Moreover, T. Hilberdink and L.Tóth in [16] derived more general asymptotic formulas, concerning the summation over $k \geq 3$ arguments: for any real $a > 0$ and for any $\epsilon > 0$ they obtained

$$\sum_{n_1,\ldots,n_k \leq x} [n_1, \ldots, n_k]^a = C_{a,k}\ x^{k(a+1)} + O_\epsilon\left(x^{k(a+1)-\frac{1}{2}+\epsilon}\right)$$

and

$$\sum_{n_1,\ldots,n_k \leq x} \left(\frac{[n_1, \ldots, n_k]}{(n_1, \ldots, n_k)}\right)^a = D_{a,k}\ x^{k(a+1)}\ + O_\epsilon\left(x^{k(a+1)-\frac{1}{2}+\epsilon}\right)$$

for some positive constants $C_{a,k}$ and $D_{a,k}$.

In this chapter we will generalize these results (in case of integer a) further and consider the arithmetic function of $k + \ell$ variables:

$$f(n_1, \ldots, n_{k+\ell}) := \left[\frac{[n_1, \ldots, n_k]^a}{(n_1, \ldots, n_k)^c}, \frac{[n_{k+1}, \ldots, n_{k+\ell}]^b}{(n_{k+1}, \ldots, n_{k+\ell})^d}\right].$$

This function satisfies

$$f(m_1 n_1, \ldots, m_{k+\ell} n_{k+\ell}) = f(m_1, \ldots, m_{k+\ell}) f(n_1, \ldots, n_{k+\ell}),$$

for any $m_1, \ldots, m_{k+\ell}, n_1, \ldots, n_{k+\ell} \in \mathbb{N}$ such that $(m_1 \ldots m_{k+\ell}, n_1 \ldots n_{k+\ell}) = 1$, i.e. f is an example of a multiplicative arithmetic function of several variables. Using the methods from [16], we will prove the following theorem:

Theorem 1. *Let $k \geq 2$, $\ell \geq 1$, $a \geq c \geq 1$ and $b \geq d \geq 0$ be fixed integers. Then for every $\epsilon > 0$ we have*

$$\sum_{n_1,\ldots,n_{k+\ell} \leq x} \left[\frac{[n_1, \ldots, n_k]^a}{(n_1, \ldots, n_k)^c}, \frac{[n_{k+1}, \ldots, n_{k+\ell}]^b}{(n_{k+1}, \ldots, n_{k+\ell})^d}\right]$$

$$= \frac{C_{k,a,c;\ell,b,d}}{(a+1)^k (b+1)^\ell} \quad x^{k(a+1)+\ell(b+1)} + O_\epsilon\left(x^{k(a+1)+\ell(b+1)-\frac{1}{2}+\epsilon}\right) \qquad (3.1)$$

and

$$\sum_{n_1,\ldots,n_{k+\ell}\le x} \frac{\left[\frac{[n_1,\ldots,n_k]^a}{(n_1,\ldots,n_k)^c}, \frac{[n_{k+1},\ldots,n_{k+\ell}]^b}{(n_{k+1},\ldots,n_{k+\ell})^d}\right]}{(n_1\ldots n_k)^a(n_{k+1}\ldots n_{k+\ell})^b} = C_{k,a,c;\ell,b,d}\, x^{k+\ell} + O_\epsilon\left(x^{k+\ell-\frac{1}{2}+\epsilon}\right), \tag{3.2}$$

where the constant $C_{k,a,c;\ell,b,d}$ is given by the Euler product

$$\prod_p \left(1-\frac{1}{p}\right)^{k+\ell} \sum_{\nu_1,\ldots,\nu_{k+\ell}=0}^{\infty} \frac{p^{\max\{(a\max - c\min)\{\nu_1,\ldots,\nu_k\},(b\max - d\min)\{\nu_{k+1},\ldots,\nu_{k+\ell}\}\}}}{p^{(a+1)(\nu_1+\cdots+\nu_k)+(b+1)(\nu_{k+1}+\cdots+\nu_{k+\ell})}}.$$

Here and through the chapter we will use the following notation:

$$(a\max - c\min)\{\nu_1,\ldots,\nu_k\} := a\cdot\max\{\nu_1,\ldots,\nu_k\} - c\cdot\min\{\nu_1,\ldots,\nu_k\}.$$

As an illustration, we obtain the following corollaries:

Corollary 1. *For every $\epsilon > 0$ we have*

$$\sum_{n_1,n_2,n_3\le x} \left[\frac{[n_1,n_2]}{(n_1,n_2)}, n_3\right] = \frac{C_{2,1,1;1,1,0}}{8}\, x^6 + O_\epsilon\left(x^{\frac{11}{2}+\epsilon}\right)$$

and

$$\sum_{n_1,n_2,n_3\le x} \frac{\left[\frac{[n_1,n_2]}{(n_1,n_2)}, n_3\right]}{n_1n_2n_3} = C_{2,1,1;1,1,0}\, x^3 + O_\epsilon\left(x^{\frac{5}{2}+\epsilon}\right)$$

where

$$C_{2,1,1;1,1,0} = \zeta(3)\zeta(4) \prod_p \left(1-\frac{3}{p^2}+\frac{3}{p^3}-\frac{2}{p^4}+\frac{1}{p^5}\right). \tag{3.3}$$

Corollary 2. *For every $\epsilon > 0$ we have*

$$\sum_{n_1,n_2,n_3\le x} \left[\frac{[n_1,n_2]^3}{(n_1,n_2)}, n_3^2\right] = \frac{C_{2,3,1;1,2,0}}{48}\, x^{11} + O_\epsilon\left(x^{\frac{21}{2}+\epsilon}\right)$$

and

$$\sum_{n_1 n_2, n_3 \le x} \frac{\left[\dfrac{[n_1,n_2]^3}{(n_1,n_2)}, n_3^2\right]}{n_1^3 n_2^3 n_3^2} = C_{2,3,1;1,2,0}\, x^3 + O_\epsilon\left(x^{\frac{5}{2}+\epsilon}\right),$$

where

$$\begin{aligned} C_{2,3,1;1,2,0} &= \zeta(3)\zeta(6)\zeta(9)\zeta(11) \\ &\times \prod_p \Bigg(1 - \frac{3}{p^2} + \frac{1}{p^3} + \frac{2}{p^4} - \frac{1}{p^5} + \frac{2}{p^6} - \frac{7}{p^7} + \frac{10}{p^8} \\ &\quad - \frac{9}{p^9} + \frac{5}{p^{10}} - \frac{1}{p^{11}} - \frac{1}{p^{12}} + \frac{5}{p^{13}} - \frac{9}{p^{14}} + \frac{10}{p^{15}} - \frac{7}{p^{16}} \\ &\quad + \frac{2}{p^{17}} - \frac{1}{p^{18}} + \frac{2}{p^{19}} + \frac{1}{p^{20}} - \frac{3}{p^{21}} + \frac{1}{p^{23}}\Bigg). \end{aligned} \tag{3.4}$$

By the method of proof in Theorem 1 we obtained the relative error of size $O(x^{-1/2+\epsilon})$. It remains as an interesting open question to determine the best possible exponent in the error term.

3.2 ***Proof of Theorem 1***

To prove this theorem we need the following lemma:

Lemma 1. *For integers $k \ge 2,\ \ell \ge 1,\ a \ge c \ge 1$ and $b \ge d \ge 0$ we have*

$$L(z_1, \dots, z_{k+\ell}) := \sum_{n_1,\dots,n_{k+\ell}=1}^{\infty} \frac{\left[\dfrac{[n_1,\dots,n_k]^a}{(n_1,\dots,n_k)^c}, \dfrac{[n_{k+1},\dots,n_{k+\ell}]^b}{(n_{k+1},\dots,n_{k+\ell})^d}\right]}{n_1^{z_1}\dots n_k^{z_k} n_{k+1}^{z_{k+1}} \dots n_{k+\ell}^{z_{k+\ell}}}$$

$$= \zeta(z_1 - a)\dots\zeta(z_k - a)\zeta(z_{k+1} - b)\dots\zeta(z_{k+\ell} - b)\, H(z_1, \dots, z_{k+\ell}), \tag{3.5}$$

where the multiple Dirichlet series $H(z_1, \dots, z_{k+\ell})$ is absolutely convergent for

$$\Re z_1, \dots, \Re z_k > a + \frac{1}{2} \qquad \textit{and} \qquad \Re z_{k+1}, \dots, \Re z_{k+\ell} > b + \frac{1}{2}. \tag{3.6}$$

Proof. Since the function

$$(n_1,\ldots,n_{k+\ell}) \longmapsto \left[\frac{[n_1,\ldots,n_k]^a}{(n_1,\ldots,n_k)^c},\ \frac{[n_{k+1},\ldots,n_{k+\ell}]^b}{(n_{k+1},\ldots,n_{k+\ell})^d}\right]$$

is a multiplicative function of $k+\ell$ variables, the multiple Dirichlet series $L(z_1,\ldots,z_{k+\ell})$ has the following Euler product expansion

$$L(z_1,\ldots,z_{k+\ell}) = \prod_p \sum_{\nu_1,\ldots,\nu_{k+\ell}=0}^{\infty} \frac{p^{\max\{(a\max-c\min)\{\nu_1,\ldots,\nu_k\},(b\max-d\min)\{\nu_{k+1},\ldots,\nu_{k+\ell}\}\}}}{p^{\nu_1 z_1+\ldots+\nu_k z_k+\nu_{k+1}z_{k+1}+\ldots+\nu_{k+\ell}z_{k+\ell}}}.$$

In each Euler's local factor, we single out the contribution of the terms for which $\nu_1+\ldots+\nu_{k+\ell}\le 1$:

$$L(z_1,\ldots,z_{k+\ell}) = \prod_p \Bigg(1+\frac{p^a}{p^{z_1}}+\ldots+\frac{p^a}{p^{z_k}}+\frac{p^b}{p^{z_{k+1}}}+\ldots+\frac{p^b}{p^{z_{k+\ell}}}+$$

$$+\sum_{\nu_1+\ldots+\nu_{k+\ell}\ge 2} \frac{p^{\max\{(a\max-c\min)\{\nu_1,\ldots,\nu_k\},(b\max-d\min)\{\nu_{k+1},\ldots,\nu_{k+\ell}\}\}}}{p^{\nu_1 z_1+\ldots+\nu_{k+\ell}z_{k+\ell}}}\Bigg). \tag{3.7}$$

Now, for $(z_1,\ldots,z_{k+\ell})$ in the region $\Re z_1,\ldots,\Re z_k \ge \delta_1 > a$, $\Re z_{k+1},\ldots,\Re z_{k+\ell} \ge \delta_2 > b$ (for some fixed $\delta_1 > a,\ \delta_2 > b$), we have that

$$\left|\frac{p^{\max\{(a\max-c\min)\{\nu_1,\ldots,\nu_k\},(b\max-d\min)\{\nu_{k+1},\ldots,\nu_{k+\ell}\}\}}}{p^{\nu_1 z_1+\ldots+\nu_{k+\ell}z_{k+\ell}}}\right|$$

$$\le \frac{p^{a(\nu_1+\ldots+\nu_k)+b(\nu_{k+1}+\ldots+\nu_{k+\ell})}}{p^{\delta_1(\nu_1+\ldots+\nu_k)+\delta_2(\nu_{k+1}+\ldots+\nu_{k+\ell})}} = \frac{1}{p^{(\delta_1-a)(\nu_1+\ldots+\nu_k)+(\delta_2-b)(\nu_{k+1}+\ldots+\nu_{k+\ell})}}.$$

Therefore, since $N_k(n) := \#\{(\nu_1,\ldots,\nu_k) \mid \nu_1+\ldots+\nu_k = n\} = \binom{n+k-1}{k-1}$, the sum over $\nu_1+\ldots+\nu_{k+\ell}\ge 2$ in (3.7) is bounded by

$$\sum_{m+n\ge 2} \frac{N_k(m)N_\ell(n)}{p^{(\delta_1-a)m+(\delta_2-b)n}} = O\left(\frac{1}{p^{2(\delta_1-a)}}+\frac{1}{p^{2(\delta_2-b)}}\right).$$

Hence for $\Re z_j > \max\{a+1,\delta_1\}$ for all $1\le j\le k$ and $\Re z_j > \max\{b+1,\delta_2\}$ for all $k+1\le j\le k+\ell$ we have

$$H(z_1,\ldots,z_{k+\ell}) := L(z_1,\ldots,z_{k+\ell})\zeta^{-1}(z_1-a)\ldots\zeta^{-1}(z_k-a)\zeta^{-1}(z_{k+1}-b)\ldots\zeta^{-1}(z_{k+\ell}-b)$$

$$= \prod_p \left(1-\frac{1}{p^{z_1-a}}\right)\ldots\left(1-\frac{1}{p^{z_k-a}}\right)\left(1-\frac{1}{p^{z_{k+1}-b}}\right)\ldots\left(1-\frac{1}{p^{z_{k+\ell}-b}}\right)$$

$$\times \left(1+\frac{1}{p^{z_1-a}}+\ldots+\frac{1}{p^{z_k-a}}+\frac{1}{p^{z_{k+1}-b}}+\ldots+\frac{1}{p^{z_{k+\ell}-b}}+O\left(\frac{1}{p^{2(\delta_1-a)}}+\frac{1}{p^{2(\delta_2-b)}}\right)\right)$$

$$=\prod_p \left(1+\ O\left(\frac{1}{p^{2(\delta_1-a)}}+\frac{1}{p^{2(\delta_2-b)}}\right)\right), \tag{3.8}$$

since the terms $\pm\frac{1}{p^{z_j-a}}$ and $\pm\frac{1}{p^{z_j-b}}$ cancel out.

The Euler product in (3.8) converges absolutely for $\delta_1 > a+\frac{1}{2}$ and $\delta_2 > b+\frac{1}{2}$. Thus the identity (3.5) holds in the product of half-planes (3.6). □

Proof (*of Theorem 1*). Let us define the multiplicative function $h(n_1,\ldots,n_{k+\ell})$ as coefficients of the multiple Dirichlet series expansion of the function $H(z_1,\ldots,z_{k+\ell})$ from Lemma 1:

$$H(z_1,\ldots,z_{k+\ell})=\sum_{n_1,\ldots,n_{k+\ell}=1}^{\infty}\frac{h(n_1,\ldots,n_{k+\ell})}{n_1^{z_1}\ldots n_{k+\ell}^{z_{k+\ell}}}.$$

From the identity (3.5) we obtain the following convolution identity between the corresponding arithmetic functions:

$$\left[\frac{[n_1,\ldots,n_k]^a}{(n_1,\ldots,n_k)^c},\frac{[n_{k+1},\ldots,n_{k+\ell}]^b}{(n_{k+1},\ldots,n_{k+\ell})^d}\right] \tag{3.9}$$

$$=\sum_{j_1d_1=n_1,\ldots,j_{k+\ell}d_{k+\ell}=n_{k+\ell}} j_1^a\ldots j_k^a\, j_{k+1}^b\ldots j_{k+\ell}^b\, h(d_1,\ldots,d_{k+\ell}).$$

By using this identity we get

$$\sum_{n_1,\ldots,n_{k+\ell}\le x}\left[\frac{[n_1,\ldots,n_k]^a}{(n_1,\ldots,n_k)^c},\ \frac{[n_{k+1},\ldots,n_{k+\ell}]^b}{(n_{k+1},\ldots,n_{k+\ell})^d}\right]$$

$$=\sum_{j_1d_1\le x,\ldots,j_{k+\ell}d_{k+\ell}\le x} j_1^a\ldots j_k^a\, j_{k+1}^b\ldots j_{k+\ell}^b\, h(d_1,\ldots,d_{k+\ell})$$

$$=\sum_{d_1,\ldots,d_{k+\ell}\le x} h(d_1,\ldots,d_{k+\ell})\sum_{j_1\le \frac{x}{d_1}} j_1^a\ldots\sum_{j_k\le\frac{x}{d_k}} j_k^a\sum_{j_{k+1}\le\frac{x}{d_{k+1}}} j_{k+1}^b\ldots\sum_{j_{k+\ell}\le\frac{x}{d_{k+\ell}}} j_{k+\ell}^b$$

$$=\sum_{d_1,\ldots,d_{k+\ell}\le x} h(d_1,\ldots,d_{k+\ell})\left(\frac{x^{a+1}}{(a+1)d_1^{a+1}}+O\left(\frac{x^a}{d_1^a}\right)\right)\ldots\left(\frac{x^{b+1}}{(b+1)d_{k+\ell}^{b+1}}+O\left(\frac{x^b}{d_{k+\ell}^b}\right)\right)$$

$$=\frac{x^{k(a+1)+\ell(b+1)}}{(a+1)^k(b+1)^\ell}\sum_{d_1,\ldots,d_{k+\ell}\le x}\frac{h(d_1,\ldots,d_{k+\ell})}{d_1^{a+1}\ldots d_k^{a+1}d_{k+1}^{b+1}\ldots d_{k+\ell}^{b+1}}+R(x), \tag{3.10}$$

where the remainder term $R(x)$ is bounded by

$$R(x) \ll \sum_{\substack{u_1,\dots,u_k\in\{a,a+1\},\\ v_1,\dots,v_\ell\in\{b,b+1\},\\ (u_1,\dots,u_k,v_1,\dots,v_\ell)\neq\\ (a+1,\dots,a+1,b+1,\dots,b+1)}} x^{u_1+\dots+u_k+v_1+\dots+v_\ell} \sum_{d_1,\dots,d_{k+\ell}\le x} \frac{|h(d_1,\dots,d_{k+\ell})|}{d_1^{u_1}\dots d_k^{u_k} d_{k+1}^{v_1}\dots d_{k+\ell}^{v_\ell}}. \tag{3.11}$$

Here the first summation is over $2^{k+\ell}-1$ $(k+\ell)$-tuples $(u_1,\dots,u_k,v_1,\dots,v_\ell)$ in which at least one u_i is a or at least one v_j is b. Let $(u_1,\dots,u_k,v_1,\dots,v_\ell)$ be one such fixed $(k+\ell)$-tuple with $s\ge 1$ u_i-coordinates equal a, for example $(u_1,\dots,u_s,u_{s+1},\dots,u_k,v_1,\dots,v_\ell)=(a,\dots,a,a+1,\dots,a+1,b+1,\dots,b+1)$. The corresponding contribution on the right hand side of (3.11) is bounded by

$$\ll x^{sa+(k-s)(a+1)+\ell(b+1)} \sum_{d_1,\dots,d_{k+\ell}\le x} \frac{|h(d_1,\dots,d_{k+\ell})|}{d_1^{a}\dots d_s^{a} d_{s+1}^{a+1}\dots d_k^{a+1} d_{k+1}^{b+1}\dots d_{k+\ell}^{b+1}}$$

$$= x^{sa+(k-s)(a+1)+\ell(b+1)} \sum_{d_1,\dots,d_{k+\ell}\le x} \frac{|h(d_1,\dots,d_{k+\ell})| d_1^{\frac12+\epsilon}\dots d_s^{\frac12+\epsilon}}{d_1^{a+\frac12+\epsilon}\dots d_s^{a+\frac12+\epsilon} d_{s+1}^{a+1}\dots d_k^{a+1} d_{k+1}^{b+1}\dots d_{k+\ell}^{b+1}}$$

$$\le x^{sa+(k-s)(a+1)+\ell(b+1)+s(\frac12+\epsilon)} \sum_{d_1,\dots,d_{k+\ell}=1}^{\infty} \frac{|h(d_1,\dots,d_{k+\ell})|}{d_1^{a+\frac12+\epsilon}\dots d_s^{a+\frac12+\epsilon} d_{s+1}^{a+1}\dots d_k^{a+1} d_{k+1}^{b+1}\dots d_{k+\ell}^{b+1}},$$

for any $\epsilon>0$. Here, the exponents $(a+\frac12+\epsilon,\dots,a+\frac12+\epsilon,a+1,\dots,a+1,b+1,\dots,\ b+1)$ belong to the region of absolute convergence (3.6) and hence, by Lemma 1 the last multiple Dirichlet series converges to a constant and we obtain the bound

$$\ll x^{k(a+1)+\ell(b+1)-\frac{s}{2}+s\epsilon}.$$

This is maximal for $s=1$. Similarly we bound the contributions in (3.11) corresponding to all other $(k+\ell)$-tuples $(u_1,\dots,u_k,v_1,\dots,v_\ell)$. Therefore we get

$$R(x) \ll x^{k(a+1)+\ell(b+1)-\frac12+\epsilon}. \tag{3.12}$$

Next, we write the sum in the main term in (3.10) as follows:

$$\sum_{d_1,\dots,d_{k+\ell}\le x} \frac{h(d_1,\dots,d_{k+\ell})}{d_1^{a+1}\dots d_k^{a+1} d_{k+1}^{b+1}\dots d_{k+\ell}^{b+1}} = \sum_{d_1,\dots,d_{k+\ell}=1}^{\infty} \frac{h(d_1,\dots,d_{k+\ell})}{d_1^{a+1}\dots d_k^{a+1} d_{k+1}^{b+1}\dots d_{k+\ell}^{b+1}}$$

$$- \sum_{\emptyset \neq I \subseteq \{1,\ldots,k+\ell\}} \sum_{\substack{d_i > x, i \in I \\ d_j \leq x, j \notin I}} \frac{h(d_1, \ldots, d_{k+\ell})}{d_1^{a+1} \ldots d_k^{a+1} d_{k+1}^{b+1} \ldots d_{k+\ell}^{b+1}}. \tag{3.13}$$

The complete multiple Dirichlet series in (3.13) converges by Lemma 1 and is equal $H(a+1, \ldots, a+1, b+1, \ldots, b+1)$. For one fixed subset I, say $I = \{1, 2, \ldots, s, k+1, k+2, \ldots, k+t\}$, $1 \leq s \leq k$, $1 \leq t \leq \ell$, the corresponding contribution in (3.13) is bounded by

$$\sum_{\substack{d_1,\ldots,d_s > x \\ d_{s+1},\ldots,d_k \leq x \\ d_{k+1},\ldots,d_{k+t} > x \\ d_{k+t+1},\ldots,d_{k+\ell} \leq x}} \frac{|h(d_1, \ldots, d_{k+\ell})|}{d_1^{a+1} \ldots d_k^{a+1} d_{k+1}^{b+1} \ldots d_{k+\ell}^{b+1}}$$

$$= \sum_{\substack{d_1,\ldots,d_s > x \\ d_{s+1},\ldots,d_k \leq x \\ d_{k+1},\ldots,d_{k+t} > x \\ d_{k+t+1},\ldots,d_{k+\ell} \leq x}} \frac{|h(d_1, \ldots, d_{k+\ell})| d_1^{-\frac{1}{2}+\epsilon} \ldots d_s^{-\frac{1}{2}+\epsilon} d_{k+1}^{-\frac{1}{2}+\epsilon} \ldots d_{k+t}^{-\frac{1}{2}+\epsilon}}{d_1^{a+\frac{1}{2}+\epsilon} \ldots d_s^{a+\frac{1}{2}+\epsilon} d_{s+1}^{a+1} \ldots d_k^{a+1} d_{k+1}^{b+\frac{1}{2}+\epsilon} \ldots d_{k+t}^{b+\frac{1}{2}+\epsilon} d_{k+t+1}^{b+1} \ldots d_{k+\ell}^{b+1}}$$

$$\leq x^{(s+t)(-\frac{1}{2}+\epsilon)} \sum_{d_1,\ldots,d_{k+\ell}=1}^{\infty} \frac{|h(d_1, \ldots, d_{k+\ell})|}{d_1^{a+\frac{1}{2}+\epsilon} \ldots d_s^{a+\frac{1}{2}+\epsilon} d_{s+1}^{a+1} \ldots d_k^{a+1} d_{k+1}^{b+\frac{1}{2}+\epsilon} \ldots d_{k+t}^{b+\frac{1}{2}+\epsilon} d_{k+t+1}^{b+1} \ldots d_{k+\ell}^{b+1}}.$$

Here, the multiple Dirichlet series converges to a constant by Lemma 1 since

$$\left(a + \frac{1}{2} + \epsilon, \ldots, a + \frac{1}{2} + \epsilon, a+1, \ldots a+1, b + \frac{1}{2} + \epsilon, \ldots, b + \frac{1}{2} + \epsilon, b+1, \ldots, b+1\right)$$

belongs to the region (3.6). For all $I \neq \emptyset$ we have that $s + t \geq 1$ and hence the total error introduced by completing the series in (3.10) is again

$$O\left(x^{k(a+1)+\ell(b+1)-\frac{1}{2}+\epsilon}\right)$$

and matches the bound for the remainder (3.12).

This proves the asymptotic formula (3.1), with the constant $C_{k,a,c;\ell,b,d} = H(a+1, \ldots, a+1, b+1, \ldots, b+1)$ which can be explicitely calculated from Lemma 1:

$$C_{k,a,c;\ell,b,d} = \prod_p \left(1 - \frac{1}{p}\right)^{k+\ell}$$

$$\times \sum_{\nu_1,\ldots,\nu_{k+\ell}=0}^{\infty} \frac{p^{\max\{(a\max - c\min)\{\nu_1,\ldots,\nu_k\},\ (b\max - d\min)\{\nu_{k+1},\ldots,\nu_{k+\ell}\}\}}}{p^{(a+1)(\nu_1+\ldots+\nu_k)+(b+1)(\nu_{k+1}+\ldots+\nu_{k+\ell})}}.$$

Proof of (3.2): From Lemma 1, i.e. from the convolution identity (3.9) we obtain

$$\frac{\left[\frac{[n_1,\ldots,n_k]^a}{(n_1,\ldots,n_k)^c},\frac{[n_{k+1},\ldots,n_{k+\ell}]^b}{(n_{k+1},\ldots,n_{k+\ell})^d}\right]}{(n_1\ldots n_k)^a(n_{k+1}\ldots n_{k+\ell})^b}=\sum_{j_1d_1=n_1,\ldots,j_{k+\ell}d_{k+\ell}=n_{k+\ell}}\frac{h(d_1,\ldots,d_{k+\ell})}{d_1^a\ldots d_k^a d_{k+1}^b\ldots d_{k+\ell}^b}.$$

Replacing this to the left-hand side of (3.2) we get

$$\sum_{n_1,\ldots,n_{k+\ell}\le x}\frac{\left[\frac{[n_1,\ldots,n_k]^a}{(n_1,\ldots,n_k)^c},\frac{[n_{k+1},\ldots,n_{k+\ell}]^b}{(n_{k+1},\ldots,n_{k+\ell})^d}\right]}{(n_1\ldots n_k)^a(n_{k+1}\ldots n_{k+\ell})^b}$$

$$=\sum_{d_1,\ldots,d_{k+\ell}\le x}\frac{h(d_1,\ldots,d_{k+\ell})}{d_1^a\ldots d_k^a d_{k+1}^b\ldots d_{k+\ell}^b}\sum_{j_1\le\frac{x}{d_1}}1\cdots\sum_{j_{k+\ell}\le\frac{x}{d_{k+\ell}}}1$$

$$=\sum_{d_1,\ldots,d_{k+\ell}\le x}\frac{h(d_1,\ldots,d_{k+\ell})}{d_1^a\ldots d_k^a d_{k+1}^b\ldots d_{k+\ell}^b}\left(\frac{x}{d_1}+O(1)\right)\ldots\left(\frac{x}{d_{k+\ell}}+O(1)\right)$$

$$=x^{k+\ell}\sum_{d_1,\ldots,d_{k+\ell}\le x}\frac{h(d_1,\ldots,d_{k+\ell})}{d_1^{a+1}\ldots d_k^{a+1}d_{k+1}^{b+1}\ldots d_{k+\ell}^{b+1}}+R_1(x),\qquad(3.14)$$

where the remainder $R_1(x)$ is bounded by

$$R_1(x)\ll\sum_{\substack{u_1,\ldots,u_{k+\ell}\in\{0,1\}\\(u_1,\ldots,u_{k+\ell})\neq(1,1,\ldots,1)}}x^{u_1+\cdots+u_{k+\ell}}\sum_{d_1,\ldots,d_{k+\ell}\le x}\frac{|h(d_1,\ldots,d_{k+\ell})|}{d_1^{u_1}\ldots d_{k+\ell}^{u_{k+\ell}}}.$$

By the same arguments leading to the bound (3.12), here we obtain

$$R_1(x)\ll x^{k+\ell-\frac{1}{2}+\epsilon}.$$

Similarly, we can complete the multiple Dirichlet series in the main term in (3.14) with the cost of the error term $O\left(x^{k+\ell-\frac{1}{2}+\epsilon}\right)$, which proves (3.2). □

3.3 Proofs of Corollaries

Proof (*of Corollary 1*). By Theorem 1, we have that

$$C_{2,1,1;1,1,0}=\prod_p\left(1-\frac{1}{p}\right)^3\sum_{a,b,c\ge0}\frac{p^{\max\{\max\{a,b\}-\min\{a,b\},\,c\}}}{p^{2(a+b+c)}}$$

$$= \prod_p \left(1 - \frac{1}{p}\right)^3 \big(S_1 + 2S_2 + 2S_3\big),$$

where

$$S_1 = \sum_{\substack{a=b\geq 0\\ c\geq 0}} \frac{1}{p^{4a+c}} = \frac{1}{1-\frac{1}{p}} \cdot \frac{1}{1-\frac{1}{p^4}},$$

$$S_2 = \sum_{\substack{a>b\geq 0\\ 0\leq c<a-b}} \frac{1}{p^{a+3b+2c}} = \frac{1}{1-\frac{1}{p^2}} \left(\sum_{a>b\geq 0} \frac{1}{p^{a+3b}} - \sum_{a>b\geq 0} \frac{1}{p^{3a+b}} \right)$$

$$= \frac{1}{1-\frac{1}{p^2}} \left(\frac{1}{p\big(1-\frac{1}{p}\big)} \sum_{b\geq 0} \frac{1}{p^{4b}} - \frac{1}{p^3\big(1-\frac{1}{p^3}\big)} \sum_{b\geq 0} \frac{1}{p^{4b}} \right)$$

$$= \frac{1}{1-\frac{1}{p}} \cdot \frac{1}{1-\frac{1}{p^4}} \cdot \frac{1}{p\big(1-\frac{1}{p^3}\big)}$$

and

$$S_3 = \sum_{\substack{a>b\geq 0\\ c\geq a-b}} \frac{1}{p^{2a+2b+c}} = \frac{1}{1-\frac{1}{p}} \sum_{a>b\geq 0} \frac{1}{p^{3a+b}}$$

$$= \frac{1}{1-\frac{1}{p}} \cdot \frac{1}{1-\frac{1}{p^4}} \cdot \frac{1}{p^3\big(1-\frac{1}{p^3}\big)}.$$

Then we obtain

$$S_1 + 2S_2 + 2S_3 = \frac{1 + \frac{2}{p} + \frac{1}{p^3}}{\big(1-\frac{1}{p}\big)\big(1-\frac{1}{p^3}\big)\big(1-\frac{1}{p^4}\big)}$$

and hence

$$C_{2,1,1;1,1,0} = \prod_p \frac{\big(1-\frac{1}{p}\big)^2\big(1+\frac{2}{p}+\frac{1}{p^3}\big)}{\big(1-\frac{1}{p^3}\big)\big(1-\frac{1}{p^4}\big)},$$

which gives (3.3). □

Proof (*of Corollary 3*). By Theorem 1, we have that

$$C_{2,3,1;1,2,0} = \prod_p \left(1 - \frac{1}{p}\right)^3 \sum_{a,b,c\geq 0} \frac{p^{\max\{3\max\{a,b\}-\min\{a,b\},\, 2c\}}}{p^{4(a+b)+3c}}$$

$$= \prod_p \left(1 - \frac{1}{p}\right)^3 \big(S_1 + S_2 + 2S_3 + 2S_4\big),$$

where

$$S_1 = \sum_{0\le a=b<c} \frac{1}{p^{8a+c}} = \frac{1}{p\left(1-\frac{1}{p}\right)} \sum_{a\ge 0} \frac{1}{p^{9a}} = \frac{1}{p\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^9}\right)},$$

$$S_2 = \sum_{0\le c\le a=b} \frac{1}{p^{6a+3c}} = \frac{1}{\left(1-\frac{1}{p^6}\right)\left(1-\frac{1}{p^9}\right)},$$

$$S_3 = \sum_{\substack{a>b\ge 0\\ 3a-b<2c}} \frac{1}{p^{4a+4b+c}}$$

$$= \frac{1}{p\left(1-\frac{1}{p}\right)} \sum_{\substack{a>b\ge 0\\ a\equiv b \pmod 2}} \frac{1}{p^{\frac{11}{2}a+\frac{7}{2}b}} + \frac{1}{p^{\frac{1}{2}}\left(1-\frac{1}{p}\right)} \sum_{\substack{a>b\ge 0\\ a\not\equiv b \pmod 2}} \frac{1}{p^{\frac{11}{2}a+\frac{7}{2}b}}$$

$$= \frac{1}{p^{12}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^{11}}\right)} \sum_{b\ge 0} \frac{1}{p^{9b}} + \frac{1}{p^{6}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^{11}}\right)} \sum_{b\ge 0} \frac{1}{p^{9b}}$$

$$= \frac{1}{\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^9}\right)\left(1-\frac{1}{p^{11}}\right)} \left(\frac{1}{p^6}+\frac{1}{p^{12}}\right)$$

and

$$S_4 = \sum_{\substack{a>b\ge 0\\ 3a-b\ge 2c\ge 0}} \frac{1}{p^{a+5b+3c}}$$

$$= \frac{1}{1-\frac{1}{p^3}} \left(\sum_{\substack{a>b\ge 0\\ a\equiv b \pmod 2}} \frac{1}{p^{a+5b}} - \frac{1}{p^3} \sum_{\substack{a>b\ge 0\\ a\equiv b \pmod 2}} \frac{1}{p^{\frac{11}{2}a+\frac{7}{2}b}} \right.$$

$$\left. + \sum_{\substack{a>b\ge 0\\ a\not\equiv b \pmod 2}} \frac{1}{p^{a+5b}} - \frac{1}{p^{\frac{3}{2}}} \sum_{\substack{a>b\ge 0\\ a\not\equiv b \pmod 2}} \frac{1}{p^{\frac{11}{2}a+\frac{7}{2}b}} \right)$$

$$= \frac{1}{1-\frac{1}{p^3}} \left(\frac{1}{p\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^6}\right)} - \frac{1}{\left(1-\frac{1}{p^9}\right)\left(1-\frac{1}{p^{11}}\right)} \left(\frac{1}{p^7}+\frac{1}{p^{14}}\right) \right).$$

Collecting everything, we obtain (3.4). □

3.4 An example with four variables

In this section we perform explicit computation of the arithmetic constant in a case with four variables.

Corollary 3. *For every* $\epsilon > 0$ *we have*

$$\sum_{n_1,n_2,n_3,n_4 \le x} \left[\frac{[n_1,n_2,n_3]^3}{(n_1,n_2,n_3)^2}, n_4^2 \right] = \frac{C_{3,3,2;1,2,0}}{192}\, x^{15} + O_\epsilon\left(x^{\frac{29}{2}+\epsilon}\right)$$

and

$$\sum_{n_1 n_2,n_3,n_4 \le x} \frac{\left[\frac{[n_1,n_2,n_3]^3}{(n_1,n_2,n_3)^2}, n_4^2 \right]}{n_1^3 n_2^3 n_3^3 n_4^2} = C_{3,3,2;1,2,0}\, x^4 + O_\epsilon\left(x^{\frac{7}{2}+\epsilon}\right),$$

where

$$C_{3,3,2;1,2,0} = \prod_p \left(1 - \frac{1}{p}\right)^4 (S_1 + S_2 + 3S_3 + 3S_4 + 3S_5 + 3S_6 + 6S_7 + 6S_8), \quad (3.15)$$

and each of the sums S_j, $1 \le j \le 8$ *is calculated explicitly below.*

Proof. By Theorem 1 we have

$$C_{3,3,2;1,2,0} = \prod_p \left(1 - \frac{1}{p}\right)^4 \sum_{\nu_1,\nu_2,\nu_3,\nu_4=0}^{\infty} \frac{p^{\max\{(3\max - 2\min)\{\nu_1,\nu_2,\nu_3\},\, 2\nu_4\}}}{p^{4(\nu_1+\nu_2+\nu_3)+3\nu_4}}.$$

We divide the inner four-fold summation into the following cases:

1. $\nu_1 = \nu_2 = \nu_3$ and $\nu_1 \ge 2\nu_4$
2. $\nu_1 = \nu_2 = \nu_3$ and $\nu_1 < 2\nu_4$
3. $\nu_1 = \nu_2 > \nu_3$ and $3\nu_1 - 2\nu_3 \ge 2\nu_4$ (and two more symmetric cases)
4. $\nu_1 = \nu_2 > \nu_3$ and $3\nu_1 - 2\nu_3 < 2\nu_4$ (and two more symmetric cases)
5. $\nu_1 = \nu_2 < \nu_3$ and $3\nu_3 - 2\nu_1 \ge 2\nu_4$ (and two more symmetric cases)

6. $\nu_1 = \nu_2 < \nu_3$ and $3\nu_3 - 2\nu_1 < 2\nu_4$ (and two more symmetric cases)

7. $\nu_1 > \nu_2 > \nu_3$ and $3\nu_1 - 2\nu_3 \geq 2\nu_4$ (and five more symmetric cases)

8. $\nu_1 > \nu_2 > \nu_3$ and $3\nu_1 - 2\nu_3 < 2\nu_4$ (and five more symmetric cases).

For a fixed prime number p, we denote with S_j the contribution to the inner summation of all the terms which satisfy j-th condition. In that notation we can write:

$$C_{3,3,2;1,2,0} = \prod_p \left(1 - \frac{1}{p}\right)^4 (S_1 + S_2 + 3S_3 + 3S_4 + 3S_5 + 3S_6 + 6S_7 + 6S_8).$$

Now, we calculate the first two series:

$$\begin{aligned} S_1 &= \sum_{\substack{\nu_1,\nu_4\geq 0\\ \nu_1\geq 2\nu_4}} \frac{p^{\max\{\nu_1,\, 2\nu_4\}}}{p^{12\nu_1+3\nu_4}} = \sum_{\substack{\nu_1,\nu_4\geq 0\\ \nu_1\geq 2\nu_4}} \frac{1}{p^{11\nu_1+3\nu_4}} \\ &= \sum_{\nu_4\geq 0} \frac{1}{p^{3\nu_4}} \sum_{\nu_1\geq 2\nu_4} \frac{1}{p^{11\nu_1}} \\ &= \frac{1}{1-\frac{1}{p^{11}}} \sum_{\nu_4\geq 0} \frac{1}{p^{25\nu_4}} = \frac{1}{\left(1-\frac{1}{p^{11}}\right)\left(1-\frac{1}{p^{25}}\right)}; \end{aligned}$$

$$\begin{aligned} S_2 &= \sum_{\substack{\nu_1,\nu_4\geq 0\\ \nu_1< 2\nu_4}} \frac{p^{\max\{\nu_1,\, 2\nu_4\}}}{p^{12\nu_1+3\nu_4}} = \sum_{\substack{\nu_1,\nu_4\geq 0\\ \nu_1< 2\nu_4}} \frac{1}{p^{12\nu_1+\nu_4}} \\ &= \sum_{\substack{k,\nu_4\geq 0\\ 2k<2\nu_4}} \frac{1}{p^{12(2k)+\nu_4}} + \sum_{\substack{k,\nu_4\geq 0\\ 2k+1<2\nu_4}} \frac{1}{p^{12(2k+1)+\nu_4}} \\ &= \sum_{k\geq 0} \frac{1}{p^{24k}} \sum_{\nu_4\geq k+1} \frac{1}{p^{\nu_4}} + \sum_{k\geq 0} \frac{1}{p^{24k+12}} \sum_{\nu_4\geq k+1} \frac{1}{p^{\nu_4}} \\ &= \left(1+\frac{1}{p^{12}}\right) \sum_{k\geq 0} \frac{1}{p^{24k}} \sum_{\nu_4\geq k+1} \frac{1}{p^{\nu_4}} \\ &= \frac{\left(1+\frac{1}{p^{12}}\right)}{1-\frac{1}{p}} \sum_{k\geq 0} \frac{1}{p^{25k+1}} = \frac{\left(1+\frac{1}{p^{12}}\right)}{p\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^{25}}\right)}. \end{aligned}$$

For the series S_3, we first make the following rearrangement:

$$S_3 = \sum_{\substack{\nu_1,\nu_3,\nu_4\geq 0\\ \nu_1>\nu_3\\ 3\nu_1-2\nu_3\geq 2\nu_4}} \frac{p^{\max\{3\nu_1-2\nu_3,\, 2\nu_4\}}}{p^{4(2\nu_1+\nu_3)+3\nu_4}} = \sum_{\substack{\nu_1,\nu_3,\nu_4\geq 0\\ \nu_1>\nu_3\\ 3\nu_1-2\nu_3\geq 2\nu_4}} \frac{1}{p^{5\nu_1+6\nu_3+3\nu_4}}$$

$$= \sum_{\substack{\nu_1,\nu_3\geq 0\\ \nu_1>\nu_3}} \frac{1}{p^{5\nu_1+6\nu_3}} \sum_{0\leq\nu_4\leq\frac{3}{2}\nu_1-\nu_3} \frac{1}{p^{3\nu_4}}$$

$$= \sum_{\substack{\nu_1,\nu_3\geq 0\\ \nu_1>\nu_3}} \frac{1}{p^{5\nu_1+6\nu_3}} \cdot \frac{1-\frac{1}{p^{3\lfloor\frac{3}{2}\nu_1-\nu_3\rfloor+3}}}{1-\frac{1}{p^3}}$$

$$= \frac{1}{1-\frac{1}{p^3}} \sum_{\substack{\nu_1,\nu_3\geq 0\\ \nu_1>\nu_3}} \frac{1}{p^{5\nu_1+6\nu_3}} - \frac{1}{1-\frac{1}{p^3}} \sum_{\substack{\nu_1,\nu_3\geq 0\\ \nu_1>\nu_3}} \frac{1}{p^{5\nu_1+6\nu_3+3\lfloor\frac{3}{2}\nu_1-\nu_3\rfloor+3}}$$

$$= \frac{1}{1-\frac{1}{p^3}} \left(\sum_{\substack{\nu_1,\nu_3\geq 0\\ \nu_1>\nu_3}} \frac{1}{p^{5\nu_1+6\nu_3}} - \frac{1}{p^3} \sum_{\substack{\nu_1,\nu_3\geq 0\\ \nu_1>\nu_3}} \frac{1}{p^{8\nu_1+3\nu_3+3\lfloor\frac{\nu_1}{2}\rfloor}} \right)$$

$$= \frac{1}{1-\frac{1}{p^3}} \left(\sum_{\nu_3\geq 0} \frac{1}{p^{6\nu_3}} \sum_{\nu_1\geq\nu_3+1} \frac{1}{p^{5\nu_1}} - \frac{1}{p^3} \sum_{\substack{k,\nu_3\geq 0\\ 2k>\nu_3}} \frac{1}{p^{19k+3\nu_3}} - \frac{1}{p^{11}} \sum_{\substack{k,\nu_3\geq 0\\ 2k+1>\nu_3}} \frac{1}{p^{19k+3\nu_3}} \right).$$

We calculate separately the three double sums:

$$\sum_{\nu_3\geq 0} \frac{1}{p^{6\nu_3}} \sum_{\nu_1\geq\nu_3+1} \frac{1}{p^{5\nu_1}} = \frac{1}{p^5\left(1-\frac{1}{p^5}\right)} \sum_{\nu_3\geq 0} \frac{1}{p^{11\nu_3}}$$

$$= \frac{1}{p^5\left(1-\frac{1}{p^5}\right)\left(1-\frac{1}{p^{11}}\right)};$$

$$\sum_{\substack{k,\nu_3\geq 0\\ 2k>\nu_3}} \frac{1}{p^{19k+3\nu_3}} = \sum_{k\geq 1} \frac{1}{p^{19k}} \cdot \frac{1-\frac{1}{p^{6k}}}{1-\frac{1}{p^3}}$$

$$= \frac{1}{1-\frac{1}{p^3}} \left(\sum_{k\geq 1} \frac{1}{p^{19k}} - \sum_{k\geq 1} \frac{1}{p^{25k}} \right) = \frac{1}{1-\frac{1}{p^3}} \left(\frac{1}{p^{19}\left(1-\frac{1}{p^{19}}\right)} - \frac{1}{p^{25}\left(1-\frac{1}{p^{25}}\right)} \right)$$

and similarly

$$\sum_{\substack{k,\nu_3\geq 0\\ 2k+1>\nu_3}} \frac{1}{p^{19k+3\nu_3}} = \frac{1}{1-\frac{1}{p^3}}\left(\frac{1}{\left(1-\frac{1}{p^{19}}\right)} - \frac{1}{p^3\left(1-\frac{1}{p^{25}}\right)}\right).$$

Hence, together we get

$$S_3 = \frac{1}{(1-\frac{1}{p^3})^2}$$

$$\cdot\left(\frac{1-\frac{1}{p^3}}{p^5\left(1-\frac{1}{p^5}\right)\left(1-\frac{1}{p^{11}}\right)} - \left(\frac{1}{p^{22}}+\frac{1}{p^{11}}\right)\frac{1}{1-\frac{1}{p^{19}}} + \left(\frac{1}{p^{28}}+\frac{1}{p^{14}}\right)\frac{1}{1-\frac{1}{p^{25}}}\right).$$

Next, we compute the contribution in the fourth case:

$$\begin{aligned}
S_4 = \sum_{\substack{\nu_1,\nu_3,\nu_4\geq 0\\ \nu_1>\nu_3\\ 3\nu_1-2\nu_3<2\nu_4}} \frac{p^{\max\{3\nu_1-2\nu_3,\, 2\nu_4\}}}{p^{4(2\nu_1+\nu_3)+3\nu_4}} &= \sum_{\substack{\nu_1,\nu_3,\nu_4\geq 0\\ \nu_1>\nu_3\\ 3\nu_1-2\nu_3<2\nu_4}} \frac{1}{p^{8\nu_1+4\nu_3+\nu_4}} \\
&= \sum_{\substack{\nu_1,\nu_3\geq 0\\ \nu_1>\nu_3}} \frac{1}{p^{8\nu_1+4\nu_3}} \sum_{\nu_4\geq\lfloor\frac{3\nu_1}{2}\rfloor-\nu_3+1} \frac{1}{p^{\nu_4}} \\
&= \frac{1}{1-\frac{1}{p}} \sum_{\substack{\nu_1,\nu_3\geq 0\\ \nu_1>\nu_3}} \frac{1}{p^{9\nu_1+\lfloor\frac{\nu_1}{2}\rfloor+3\nu_3+1}} \\
&= \frac{1}{p\left(1-\frac{1}{p}\right)} \sum_{\nu_1\geq 1} \frac{1}{p^{9\nu_1+\lfloor\frac{\nu_1}{2}\rfloor}} \sum_{0\leq\nu_3\leq\nu_1-1} \frac{1}{p^{3\nu_3}} \\
&= \frac{1}{p\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^3}\right)}\left(\sum_{\nu_1\geq 1}\frac{1}{p^{9\nu_1+\lfloor\frac{\nu_1}{2}\rfloor}} - \sum_{\nu_1\geq 1}\frac{1}{p^{12\nu_1+\lfloor\frac{\nu_1}{2}\rfloor}}\right).
\end{aligned}$$

We calculate separately the sums for $\nu_1 = 2k$ and for $\nu_1 = 2k+1$:

$$\begin{aligned}
\sum_{\nu_1\geq 1}\frac{1}{p^{9\nu_1+\lfloor\frac{\nu_1}{2}\rfloor}} &= \sum_{k\geq 1}\frac{1}{p^{19k}} + \sum_{k\geq 0}\frac{1}{p^{19k+9}} \\
&= \left(\frac{1}{p^{19}}+\frac{1}{p^9}\right)\frac{1}{\left(1-\frac{1}{p^{19}}\right)}
\end{aligned}$$

and similarly we obtain

$$\sum_{\nu_1\geq 1}\frac{1}{p^{12\nu_1+\lfloor\frac{\nu_1}{2}\rfloor}}=\left(\frac{1}{p^{25}}+\frac{1}{p^{12}}\right)\frac{1}{\left(1-\frac{1}{p^{25}}\right)}.$$

Therefore we have

$$S_4=\frac{1}{p\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^3}\right)}\left(\frac{\left(\frac{1}{p^{19}}+\frac{1}{p^9}\right)}{\left(1-\frac{1}{p^{19}}\right)}-\frac{\left(\frac{1}{p^{25}}+\frac{1}{p^{12}}\right)}{\left(1-\frac{1}{p^{25}}\right)}\right).$$

The fifth series is

$$S_5=\sum_{\substack{\nu_1,\nu_3,\nu_4\geq 0\\ \nu_1<\nu_3\\ 3\nu_3-2\nu_1\geq 2\nu_4}}\frac{p^{\max\{3\nu_3-2\nu_1,\,2\nu_4\}}}{p^{4(2\nu_1+\nu_3)+3\nu_4}}=\sum_{\substack{\nu_1,\nu_3,\nu_4\geq 0\\ \nu_1<\nu_3\\ 3\nu_3-2\nu_1\geq 2\nu_4}}\frac{1}{p^{10\nu_1+\nu_3+3\nu_4}}$$

$$=\sum_{\substack{\nu_1,\nu_3\geq 0\\ \nu_1<\nu_3}}\frac{1}{p^{10\nu_1+\nu_3}}\sum_{0\leq\nu_4\leq\frac{3}{2}\nu_3-\nu_1}\frac{1}{p^{3\nu_4}}$$

$$=\sum_{\substack{\nu_1,\nu_3\geq 0\\ \nu_1<\nu_3}}\frac{1}{p^{10\nu_1+\nu_3}}\cdot\frac{1-\frac{1}{p^{3\lfloor\frac{3}{2}\nu_3-\nu_1\rfloor+3}}}{1-\frac{1}{p^3}}$$

$$=\frac{1}{1-\frac{1}{p^3}}\sum_{\substack{\nu_1,\nu_3\geq 0\\ \nu_1<\nu_3}}\frac{1}{p^{10\nu_1+\nu_3}}-\frac{1}{1-\frac{1}{p^3}}\sum_{\substack{\nu_1,\nu_3\geq 0\\ \nu_1<\nu_3}}\frac{1}{p^{10\nu_1+\nu_3+3\lfloor\frac{3}{2}\nu_3-\nu_1\rfloor+3}}$$

$$=\frac{1}{1-\frac{1}{p^3}}\left(\sum_{\substack{\nu_1,\nu_3\geq 0\\ \nu_1<\nu_3}}\frac{1}{p^{10\nu_1+\nu_3}}-\frac{1}{p^3}\sum_{\substack{\nu_1,\nu_3\geq 0\\ \nu_1<\nu_3}}\frac{1}{p^{7\nu_1+4\nu_3+3\lfloor\frac{\nu_3}{2}\rfloor}}\right)$$

$$=\frac{1}{1-\frac{1}{p^3}}\left(\sum_{\nu_1\geq 0}\frac{1}{p^{10\nu_1}}\sum_{\nu_3\geq\nu_1+1}\frac{1}{p^{\nu_3}}-\frac{1}{p^3}\sum_{\substack{k,\nu_1\geq 0\\ 2k>\nu_1}}\frac{1}{p^{11k+7\nu_1}}-\frac{1}{p^7}\sum_{\substack{k,\nu_1\geq 0\\ 2k+1>\nu_1}}\frac{1}{p^{11k+7\nu_1}}\right).$$

We calculate separately the three double sums inside the bracket:

$$\sum_{\nu_1\geq 0}\frac{1}{p^{10\nu_1}}\sum_{\nu_3\geq\nu_1+1}\frac{1}{p^{\nu_3}}=\frac{1}{p\left(1-\frac{1}{p}\right)}\sum_{\nu_1\geq 0}\frac{1}{p^{11\nu_1}}$$

$$= \frac{1}{p\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^{11}}\right)};$$

$$\sum_{\substack{k,\nu_1\geq 0\\ 2k>\nu_1}} \frac{1}{p^{11k+7\nu_1}} = \sum_{k\geq 1} \frac{1}{p^{11k}} \cdot \frac{1-\frac{1}{p^{14k}}}{1-\frac{1}{p^7}}$$

$$= \frac{1}{1-\frac{1}{p^7}} \left(\sum_{k\geq 1} \frac{1}{p^{11k}} - \sum_{k\geq 1} \frac{1}{p^{25k}} \right) = \frac{1}{1-\frac{1}{p^7}} \left(\frac{1}{p^{11}\left(1-\frac{1}{p^{11}}\right)} - \frac{1}{p^{25}\left(1-\frac{1}{p^{25}}\right)} \right)$$

and similarly

$$\sum_{\substack{k,\nu_1\geq 0\\ 2k+1>\nu_1}} \frac{1}{p^{11k+7\nu_1}} = \frac{1}{1-\frac{1}{p^7}} \left(\frac{1}{\left(1-\frac{1}{p^{11}}\right)} - \frac{1}{p^{7}\left(1-\frac{1}{p^{25}}\right)} \right).$$

Hence, together we get

$$S_5 = \frac{1}{(1-\frac{1}{p^3})(1-\frac{1}{p^7})}$$

$$\times \left(\frac{1-\frac{1}{p^7}}{p\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^{11}}\right)} - \left(\frac{1}{p^{14}}+\frac{1}{p^7}\right)\frac{1}{1-\frac{1}{p^{11}}} + \left(\frac{1}{p^{28}}+\frac{1}{p^{14}}\right)\frac{1}{1-\frac{1}{p^{25}}} \right).$$

Next, we compute the contribution of the multiple series of type six:

$$S_6 = \sum_{\substack{\nu_1,\nu_3,\nu_4\geq 0\\ \nu_1<\nu_3\\ 3\nu_3-2\nu_1<2\nu_4}} \frac{p^{\max\{3\nu_3-2\nu_1,\, 2\nu_4\}}}{p^{4(2\nu_1+\nu_3)+3\nu_4}} = \sum_{\substack{\nu_1,\nu_3,\nu_4\geq 0\\ \nu_1<\nu_3\\ 3\nu_3-2\nu_1<2\nu_4}} \frac{1}{p^{8\nu_1+4\nu_3+\nu_4}}$$

$$= \sum_{\substack{\nu_1,\nu_3\geq 0\\ \nu_1<\nu_3}} \frac{1}{p^{8\nu_1+4\nu_3}} \sum_{\nu_4\geq \lfloor\frac{3\nu_3}{2}\rfloor-\nu_1+1} \frac{1}{p^{\nu_4}}$$

$$= \frac{1}{1-\frac{1}{p}} \sum_{\substack{\nu_1,\nu_3\geq 0\\ \nu_1<\nu_3}} \frac{1}{p^{7\nu_1+5\nu_3+\lfloor\frac{\nu_3}{2}\rfloor+1}}$$

$$= \frac{1}{p\left(1-\frac{1}{p}\right)} \sum_{\nu_3 \geq 1} \frac{1}{p^{5\nu_3 + \lfloor \frac{\nu_3}{2} \rfloor}} \sum_{0 \leq \nu_1 \leq \nu_3 - 1} \frac{1}{p^{7\nu_1}}$$

$$= \frac{1}{p\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^7}\right)} \left(\sum_{\nu_3 \geq 1} \frac{1}{p^{5\nu_3 + \lfloor \frac{\nu_3}{2} \rfloor}} - \sum_{\nu_3 \geq 1} \frac{1}{p^{12\nu_3 + \lfloor \frac{\nu_3}{2} \rfloor}} \right).$$

In the ν_3-sums we separate the terms for $\nu_3 = 2k$ and for $\nu_3 = 2k+1$:

$$\sum_{\nu_3 \geq 1} \frac{1}{p^{5\nu_3 + \lfloor \frac{\nu_3}{2} \rfloor}} = \sum_{k \geq 1} \frac{1}{p^{11k}} + \sum_{k \geq 0} \frac{1}{p^{11k+5}}$$

$$= \left(\frac{1}{p^{11}} + \frac{1}{p^5} \right) \frac{1}{\left(1 - \frac{1}{p^{11}}\right)}$$

and similarly we obtain

$$\sum_{\nu_3 \geq 1} \frac{1}{p^{12\nu_3 + \lfloor \frac{\nu_3}{2} \rfloor}} = \left(\frac{1}{p^{25}} + \frac{1}{p^{12}} \right) \frac{1}{\left(1 - \frac{1}{p^{25}}\right)}.$$

Therefore we get

$$S_6 = \frac{1}{p\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^7}\right)} \left(\frac{\left(\frac{1}{p^{11}} + \frac{1}{p^5}\right)}{\left(1 - \frac{1}{p^{11}}\right)} - \frac{\left(\frac{1}{p^{25}} + \frac{1}{p^{12}}\right)}{\left(1 - \frac{1}{p^{25}}\right)} \right).$$

The seventh series is four-fold:

$$S_7 = \sum_{\substack{\nu_1,\nu_2,\nu_3,\nu_4 \geq 0 \\ \nu_1 > \nu_2 > \nu_3 \\ 3\nu_1 - 2\nu_3 \geq 2\nu_4}} \frac{p^{\max\{3\nu_1 - 2\nu_3,\, 2\nu_4\}}}{p^{4(\nu_1+\nu_2+\nu_3)+3\nu_4}} = \sum_{\substack{\nu_1,\nu_2,\nu_3,\nu_4 \geq 0 \\ \nu_1 > \nu_2 > \nu_3 \\ 3\nu_1 - 2\nu_3 \geq 2\nu_4}} \frac{1}{p^{\nu_1 + 4\nu_2 + 6\nu_3 + 3\nu_4}}$$

$$= \sum_{\substack{\nu_1,\nu_2,\nu_3 \geq 0 \\ \nu_1 > \nu_2 > \nu_3}} \frac{1}{p^{\nu_1 + 4\nu_2 + 6\nu_3}} \sum_{0 \leq \nu_4 \leq \lfloor \frac{3\nu_1}{2} \rfloor - \nu_3} \frac{1}{p^{3\nu_4}}$$

$$= \frac{1}{1 - \frac{1}{p^3}} \left(\sum_{\substack{\nu_1,\nu_2,\nu_3 \geq 0 \\ \nu_1 > \nu_2 > \nu_3}} \frac{1}{p^{\nu_1 + 4\nu_2 + 6\nu_3}} - \sum_{\substack{\nu_1,\nu_2,\nu_3 \geq 0 \\ \nu_1 > \nu_2 > \nu_3}} \frac{1}{p^{4\nu_1 + 3\lfloor \frac{\nu_1}{2} \rfloor + 4\nu_2 + 3\nu_3 + 3}} \right).$$

The first sum is easy to compute:

$$\sum_{\substack{\nu_1,\nu_2,\nu_3\geq 0\\ \nu_1>\nu_2>\nu_3}} \frac{1}{p^{\nu_1+4\nu_2+6\nu_3}} = \sum_{\substack{\nu_2,\nu_3\geq 0\\ \nu_2>\nu_3}} \frac{1}{p^{4\nu_2+6\nu_3}} \sum_{\nu_1\geq\nu_2+1} \frac{1}{p^{\nu_1}}$$

$$= \frac{1}{p\left(1-\frac{1}{p}\right)} \sum_{\substack{\nu_2,\nu_3\geq 0\\ \nu_2>\nu_3}} \frac{1}{p^{5\nu_2+6\nu_3}}$$

$$= \frac{1}{p^6\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^5}\right)} \sum_{\nu_3\geq 0} \frac{1}{p^{11\nu_3}}$$

$$= \frac{1}{p^6\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^5}\right)\left(1-\frac{1}{p^{11}}\right)}.$$

The second sum is

$$\frac{1}{p^3} \sum_{\substack{\nu_1,\nu_2\geq 1\\ \nu_1>\nu_2}} \frac{1}{p^{4\nu_1+3\lfloor\frac{\nu_1}{2}\rfloor+4\nu_2}} \sum_{0\leq\nu_3\leq\nu_2-1} \frac{1}{p^{3\nu_3}}$$

$$= \frac{1}{p^3\left(1-\frac{1}{p^3}\right)} \left(\sum_{\substack{\nu_1,\nu_2\geq 1\\ \nu_1>\nu_2}} \frac{1}{p^{4\nu_1+3\lfloor\frac{\nu_1}{2}\rfloor+4\nu_2}} - \sum_{\substack{\nu_1,\nu_2\geq 1\\ \nu_1>\nu_2}} \frac{1}{p^{4\nu_1+3\lfloor\frac{\nu_1}{2}\rfloor+7\nu_2}} \right) \tag{3.16}$$

The first sum inside the bracket in (3.16) can be computed as follows:

$$\sum_{\substack{\nu_1,\nu_2\geq 1\\ \nu_1>\nu_2}} \frac{1}{p^{4\nu_1+3\lfloor\frac{\nu_1}{2}\rfloor+4\nu_2}} = \sum_{\nu_1\geq 2} \frac{1}{p^{4\nu_1+3\lfloor\frac{\nu_1}{2}\rfloor}} \sum_{1\leq\nu_2\leq\nu_1-1} \frac{1}{p^{4\nu_2}}$$

$$= \frac{1}{p^4\left(1-\frac{1}{p^4}\right)} \left(\sum_{\nu_1\geq 2} \frac{1}{p^{4\nu_1+3\lfloor\frac{\nu_1}{2}\rfloor}} - p^4 \sum_{\nu_1\geq 2} \frac{1}{p^{8\nu_1+3\lfloor\frac{\nu_1}{2}\rfloor}} \right)$$

$$= \frac{1}{p^4\left(1-\frac{1}{p^4}\right)} \left(\left(1+\frac{1}{p^4}\right) \sum_{k\geq 1} \frac{1}{p^{11k}} - \left(p^4+\frac{1}{p^4}\right) \sum_{k\geq 1} \frac{1}{p^{19k}} \right)$$

$$= \frac{1}{p^{15}\left(1-\frac{1}{p^4}\right)} \left(\frac{1+\frac{1}{p^4}}{1-\frac{1}{p^{11}}} - \frac{\frac{1}{p^4}+\frac{1}{p^{12}}}{1-\frac{1}{p^{19}}} \right).$$

In the same way, the second sum in (3.16) is equal to

$$\sum_{\substack{\nu_1,\nu_2\geq 1\\ \nu_1>\nu_2}} \frac{1}{p^{4\nu_1+3\lfloor\frac{\nu_1}{2}\rfloor+7\nu_2}} =$$

$$= \frac{1}{p^{18}\left(1-\frac{1}{p^7}\right)}\left(\frac{1+\frac{1}{p^4}}{1-\frac{1}{p^{11}}} - \frac{\frac{1}{p^7}+\frac{1}{p^{18}}}{1-\frac{1}{p^{25}}}\right).$$

Taking everything together, the seventh series is equal to

$$S_7 = \frac{1}{p^6\left(1-\frac{1}{p^3}\right)^2}\left[\frac{1-\frac{1}{p^3}}{\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^5}\right)\left(1-\frac{1}{p^{11}}\right)}\right.$$

$$-\frac{1+\frac{1}{p^4}}{p^{12}\left(1-\frac{1}{p^4}\right)\left(1-\frac{1}{p^{11}}\right)} + \frac{1+\frac{1}{p^8}}{p^{16}\left(1-\frac{1}{p^4}\right)\left(1-\frac{1}{p^{19}}\right)}$$

$$\left.-\frac{1+\frac{1}{p^4}}{p^{15}\left(1-\frac{1}{p^7}\right)\left(1-\frac{1}{p^{11}}\right)} + \frac{1+\frac{1}{p^{11}}}{p^{22}\left(1-\frac{1}{p^7}\right)\left(1-\frac{1}{p^{25}}\right)}\right].$$

We compute the last sum:

$$S_8 = \sum_{\substack{\nu_1,\nu_2,\nu_3,\nu_4\geq 0\\ \nu_1>\nu_2>\nu_3\\ 3\nu_1-2\nu_3<2\nu_4}} \frac{p^{\max\{3\nu_1-2\nu_3,\ 2\nu_4\}}}{p^{4(\nu_1+\nu_2+\nu_3)+3\nu_4}} = \sum_{\substack{\nu_1,\nu_2,\nu_3,\nu_4\geq 0\\ \nu_1>\nu_2>\nu_3\\ 3\nu_1-2\nu_3<2\nu_4}} \frac{1}{p^{4\nu_1+4\nu_2+4\nu_3+\nu_4}}$$

$$= \sum_{\substack{\nu_1,\nu_2,\nu_3\geq 0\\ \nu_1>\nu_2>\nu_3}} \frac{1}{p^{4\nu_1+4\nu_2+4\nu_3}} \sum_{\nu_4\geq\lfloor\frac{3\nu_1}{2}\rfloor-\nu_3+1} \frac{1}{p^{\nu_4}}$$

$$= \frac{1}{p\left(1-\frac{1}{p}\right)} \sum_{\substack{\nu_1,\nu_2,\nu_3\geq 0\\ \nu_1>\nu_2>\nu_3}} \frac{1}{p^{5\nu_1+\lfloor\frac{\nu_1}{2}\rfloor+4\nu_2+3\nu_3}}$$

$$= \frac{1}{p\left(1-\frac{1}{p}\right)} \sum_{\substack{\nu_1,\nu_2\geq 1\\ \nu_1>\nu_2}} \frac{1}{p^{5\nu_1+\lfloor\frac{\nu_1}{2}\rfloor+4\nu_2}} \sum_{0\leq\nu_3\leq\nu_2-1} \frac{1}{p^{3\nu_3}}$$

$$= \frac{1}{p\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^3}\right)}\left(\sum_{\substack{\nu_1,\nu_2\geq 1\\ \nu_1>\nu_2}}\frac{1}{p^{5\nu_1+\lfloor\frac{\nu_1}{2}\rfloor+4\nu_2}}-\sum_{\substack{\nu_1,\nu_2\geq 1\\ \nu_1>\nu_2}}\frac{1}{p^{5\nu_1+\lfloor\frac{\nu_1}{2}\rfloor+7\nu_2}}\right). \tag{3.17}$$

The first sum inside the bracket in (3.17) can be computed as follows:

$$\sum_{\substack{\nu_1,\nu_2\geq 1\\ \nu_1>\nu_2}}\frac{1}{p^{5\nu_1+\lfloor\frac{\nu_1}{2}\rfloor+4\nu_2}} = \sum_{\nu_1\geq 2}\frac{1}{p^{5\nu_1+\lfloor\frac{\nu_1}{2}\rfloor}}\sum_{1\leq\nu_2\leq\nu_1-1}\frac{1}{p^{4\nu_2}}$$

$$= \frac{1}{p^4\left(1-\frac{1}{p^4}\right)}\left(\sum_{\nu_1\geq 2}\frac{1}{p^{5\nu_1+\lfloor\frac{\nu_1}{2}\rfloor}} - p^4\sum_{\nu_1\geq 2}\frac{1}{p^{9\nu_1+\lfloor\frac{\nu_1}{2}\rfloor}}\right)$$

$$= \frac{1}{p^4\left(1-\frac{1}{p^4}\right)}\left(\left(1+\frac{1}{p^5}\right)\sum_{k\geq 1}\frac{1}{p^{11k}} - \left(p^4+\frac{1}{p^5}\right)\sum_{k\geq 1}\frac{1}{p^{19k}}\right)$$

$$= \frac{1}{p^{15}\left(1-\frac{1}{p^4}\right)}\left(\frac{1+\frac{1}{p^5}}{1-\frac{1}{p^{11}}} - \frac{\frac{1}{p^4}+\frac{1}{p^{13}}}{1-\frac{1}{p^{19}}}\right).$$

In the same way, the second sum in (3.17) is equal to

$$\sum_{\substack{\nu_1,\nu_2\geq 1\\ \nu_1>\nu_2}}\frac{1}{p^{5\nu_1+\lfloor\frac{\nu_1}{2}\rfloor+7\nu_2}} =$$

$$= \frac{1}{p^{18}\left(1-\frac{1}{p^7}\right)}\left(\frac{1+\frac{1}{p^5}}{1-\frac{1}{p^{11}}} - \frac{\frac{1}{p^7}+\frac{1}{p^{19}}}{1-\frac{1}{p^{25}}}\right).$$

Therefore, we get

$$S_8 = \frac{1}{p^{16}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^3}\right)}\left[\frac{1+\frac{1}{p^5}}{\left(1-\frac{1}{p^4}\right)\left(1-\frac{1}{p^{11}}\right)} - \frac{1+\frac{1}{p^9}}{p^4\left(1-\frac{1}{p^4}\right)\left(1-\frac{1}{p^{19}}\right)}\right.$$
$$\left. - \frac{1+\frac{1}{p^5}}{p^3\left(1-\frac{1}{p^7}\right)\left(1-\frac{1}{p^{11}}\right)} + \frac{1+\frac{1}{p^{12}}}{p^{10}\left(1-\frac{1}{p^7}\right)\left(1-\frac{1}{p^{25}}\right)}\right].$$

Finally, adding everything together, we get the constant (3.15).

□

4. ON SOME MULTIVARIATE SUMMATORY FUNCTIONS OF THE EULER PHI-FUNCTION

In this chapter we obtain an asymptotic formula with a power saving error term for the summation function of Euler phi-function evaluated at iterated and generalized least common multiples of four integer variables.

4.1 *Introduction*

In this chapter we denote by $[n_1, \ldots, n_k]$ the least common multiple and by $(n_1, \ldots, n_k)$ the greatest common divisor of positive integers $n_1, \ldots, n_k$. In [9], Diaconis and Erdős obtained asymptotic formulas for summatory functions

$$\sum_{m,n\le x} (m,n) \qquad \text{and} \qquad \sum_{m,n\le x} [m,n]$$

of the greatest common divisor and the least common multiple. More recently, Hilberdink in [14] investigated in more details the arithmetic function $\circ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, defined by $m \circ n := \frac{[m,n]}{(m,n)}$, which has several very interesting properties. For example, the set of squarefree positive integers is an abelian group with respect to the operation $\circ$. Moreover, for any squarefree integer $k \in \mathbb{N}$, the set $D(k)$ of all divisors of k is a finite abelian group under the restriction of $\circ$ on $D(k)$. Hilberdink investigated in depth discrete Fourier analysis and multiplicative functions on these finite groups $D(k)$. One particularly interesting feature is that the restriction of Möbius function μ on $D(k)$ is one of the characters of this group.

Quotients $\frac{[m,n]}{(m,n)}$ of the least common multiple and the greatest common divisor of integers m and n appear in many papers in linear algebra (dealing with 'arithmetical matrices') and in number theory, see for example [10, 11, 13, 15]. Recently, T. Hilberdink and L.Tóth in [16] considered the problem of establishing an asymptotic formula for the summation function of $\frac{[m,n]}{(m,n)}$ and obtained the formula

$$\sum_{m,n\le x} \frac{[m,n]}{(m,n)} = \frac{\pi^2}{60}\, x^4 + O(x^3 \log x).$$

Moreover, the authors in [16] derived more general asymptotic formulas, where the analogous summation is taken over $k \ge 3$ arguments. For an arithmetic function f from a suitable class of multiplicative functions, the authors of [16] obtained the asymptotic formulas for

$$\sum_{n_1,\ldots,n_k\le x} f\left([n_1,\ldots,n_k]\right) \qquad \text{and} \qquad \sum_{n_1,\ldots,n_k\le x} f\left(\frac{[n_1,\ldots,n_k]}{(n_1,\ldots,n_k)}\right),$$

with the power saving of $O(x^{1/2-\epsilon})$ in the error terms in both cases.

We considered in [1] further summatory function for the following "generalized" least common multiple $\left[\frac{[n_1,\ldots,n_k]^a}{(n_1,\ldots,n_k)^c}, \frac{[n_{k+1},\ldots,n_{k+\ell}]^b}{(n_{k+1},\ldots,n_{k+\ell})^d}\right]$, for integers $a \ge c \ge 1$ and $b \ge d \ge 0$, which is a multiplicative function of $k+\ell$ variables. Our goal in this chapter is to give similar generalization for the summation of Euler phi-function φ, where for simplicity of notation, we restrict ourselves to the case $k = \ell = 2$.

Theorem 2. *For integers $a,b,c,d \ge 0$, $a,b \ge 1$, $a \ge c$, $b \ge d$ and for any $0 < \epsilon < \frac{1}{2}$ we have*

$$\sum_{n_1,n_2,n_3,n_4\le x} \varphi\left(\left[\frac{[n_1,n_2]^a}{(n_1,n_2)^c}, \frac{[n_3,n_4]^b}{(n_3,n_4)^d}\right]\right)$$
$$= \frac{C_{a,c;b,d}}{(a+1)^2(b+1)^2}\, x^{2a+2b+4} + O_\epsilon\left(x^{2a+2b+\frac{7}{2}+\epsilon}\right),$$

where the implied constant depends only on ϵ and the constant $C_{a,c;b,d}$ is given by the Euler product

$$\prod_p \left(1-\frac{1}{p}\right)^4 \sum_{\nu_1,\nu_2,\nu_3,\nu_4=0}^{\infty} \frac{\varphi\left(p^{\max\{(a\max - c\min)\{\nu_1,\nu_2\},(b\max - d\min)\{\nu_3,\nu_4\}\}}\right)}{p^{(a+1)(\nu_1+\nu_2)+(b+1)(\nu_3+\nu_4)}}.$$

Here and through the thesis, $(a\max - c\min)\{\nu_1,\nu_2\}$ denotes $a\cdot\max\{\nu_1,\nu_2\} - c\cdot\min\{\nu_1,\nu_2\}$. We recall that φ is a multiplicative function which is on prime powers given by $\varphi(p^a) = p^a - p^{a-1}$. Because of multiplicativity of φ, the function $(n_1, n_2, n_3, n_4) \mapsto \varphi\left(\left[\frac{[n_1,n_2]^a}{(n_1,n_2)^c}, \frac{[n_3,n_4]^b}{(n_3,n_4)^d}\right]\right)$ will be a multiplicative function of 4 variables, enabling us to adapt the method from [16]. We recall that a function $f : \mathbb{N}^4 \to \mathbb{C}$ is multiplicative if it satisfies

$$f(m_1n_1, m_2n_2, m_3n_3, m_4n_4) = f(m_1, m_2, m_3, m_4)f(n_1, n_2, n_3, n_4)$$

whenever $(m_1m_2m_3m_4, n_1n_2n_3n_4) = 1$.

4.2 *Proof of Theorem 2*

To prove this theorem we need the following lemma:

Lemma 2. *For integers $a, b, c, d \geq 0$, $a, b \geq 1$, $a \geq c$, $b \geq d$ and complex numbers $z_j, 1 \leq j \leq 4$ such that*

$$\Re z_1, \Re z_2 > a + \frac{1}{2} \qquad \text{and} \qquad \Re z_3, \Re z_4 > b + \frac{1}{2} \tag{4.1}$$

we have

$$L(z_1, z_2, z_3, z_4) := \sum_{n_1,n_2,n_3,n_4=1}^{\infty} \frac{\varphi\left(\left[\frac{[n_1,n_2]^a}{(n_1,n_2)^c}, \frac{[n_3,n_4]^b}{(n_3,n_4)^d}\right]\right)}{n_1^{z_1} n_2^{z_2} n_3^{z_3} n_4^{z_4}}$$

$$= \zeta(z_1 - a)\zeta(z_2 - a)\zeta(z_3 - b)\zeta(z_4 - b)H(z_1, z_2, z_3, z_4), \tag{4.2}$$

where $H(z_1, z_2, z_3, z_4)$ is a certain multiple Dirichlet series defined in the proof and absolutely convergent in the region (4.1).

Proof. Because of the multiplicativity of the function

$$(n_1, n_2, n_3, n_4) \longmapsto \varphi\left(\left[\frac{[n_1, n_2]^a}{(n_1, n_2)^c}, \frac{[n_3, n_4]^b}{(n_3, n_4)^d}\right]\right),$$

by [29, Proposition 11] the multiple Dirichlet series $L(z_1,\ z_2, z_3, z_4)$ has the following Euler product expansion:

$$L(z_1, z_2, z_3, z_4) = \prod_p \sum_{\nu_1,\nu_2,\nu_3,\nu_4=0}^{\infty} \frac{\varphi\left(p^{\max\{(a\max - c\min)\{\nu_1,\nu_2\},(b\max - d\min)\{\nu_3,\nu_4\}\}}\right)}{p^{\nu_1 z_1 + \nu_2 z_2 + \nu_3 z_3 + \nu_4 z_4}}.$$

In each Euler's factor corresponding to a prime p, we single out the contribution of the terms for which $\nu_1+\nu_2+\nu_3+\nu_4 \le 1$:

$$L(z_1,z_2,z_3,z_4) = \prod_p \left(1 + \frac{p^a - p^{a-1}}{p^{z_1}} + \frac{p^a - p^{a-1}}{p^{z_2}} + \frac{p^b - p^{b-1}}{p^{z_3}} + \frac{p^b - p^{b-1}}{p^{z_4}}\right.$$

$$\left. + \sum_{\substack{\nu_1,\nu_2,\nu_3,\nu_4 \ge 0 \\ \nu_1+\nu_2+\nu_3+\nu_4 \ge 2}} \frac{\varphi\left(p^{\max\{(a\max - c\min)\{\nu_1,\nu_2\},(b\max - d\min)\{\nu_3,\nu_4\}\}}\right)}{p^{\nu_1 z_1+\nu_2 z_2+\nu_3 z_3+\nu_4 z_4}}\right). \qquad (4.3)$$

Next, for fixed $\delta_1 > a$ and $\delta_2 > b$, in the region $\Re z_1, \Re z_2 \ge \delta_1 > a$ and $\Re z_3, \Re z_4 \ge \delta_2 > b$ we have that

$$\left|\frac{\varphi\left(p^{\max\{(a\max - c\min)\{\nu_1,\nu_2\},(b\max - d\min)\{\nu_3,\nu_4\}\}}\right)}{p^{\nu_1 z_1+\nu_2 z_2+\nu_3 z_3+\nu_4 z_4}}\right|$$
$$\le \frac{p^{a(\nu_1+\nu_2)+b(\nu_3+\nu_4)}}{p^{\delta_1(\nu_1+\nu_2)+\delta_2(\nu_3+\nu_4)}} = \frac{1}{p^{(\delta_1-a)(\nu_1+\nu_2)+(\delta_2-b)(\nu_3+\nu_4)}}.$$

Since the number of solutions of $\nu_1+\nu_2 = m$ in nonnegative integers ν_1,ν_2 is $m+1$, the sum over $\nu_1+\nu_2+\nu_3+\nu_4 \ge 2$ in equation (4.3) is bounded by

$$\sum_{m+n\ge 2} \frac{(m+1)(n+1)}{p^{(\delta_1-a)m+(\delta_2-b)n}} = O\left(\frac{1}{p^{2(\delta_1-a)}} + \frac{1}{p^{2(\delta_2-b)}}\right).$$

Now, in the region $\Re z_1, \Re z_2 > \max\{\delta_1, a+1\}$ and $\Re z_3, \Re z_4 > \max\{\delta_2, b+1\}$ we can define the function

$$H(z_1,z_2,z_3,z_4) := \frac{L(z_1,z_2,z_3,z_4)}{\zeta(z_1-a)\zeta(z_2-a)\zeta(z_3-b)\zeta(z_4-b)},$$

which in this region has the following Euler product decomposition:

$$H(z_1,z_2,z_3,z_4) = \prod_p \left(1-\frac{1}{p^{z_1-a}}\right)\left(1-\frac{1}{p^{z_2-a}}\right)\left(1-\frac{1}{p^{z_3-b}}\right)\left(1-\frac{1}{p^{z_4-b}}\right)$$
$$\times \left(1 + \frac{1}{p^{z_1-a}} - \frac{1}{p^{z_1-a+1}} + \frac{1}{p^{z_2-a}} - \frac{1}{p^{z_2-a+1}} + \frac{1}{p^{z_3-b}} - \frac{1}{p^{z_3-b+1}}\right.$$
$$\left. + \frac{1}{p^{z_4-b}} - \frac{1}{p^{z_4-b+1}} + O\left(\frac{1}{p^{2(\delta_1-a)}} + \frac{1}{p^{2(\delta_2-b)}}\right)\right)$$

$$= \prod_p \left(1 + O\left(\frac{1}{p^{\delta_1 - a+1}} + \frac{1}{p^{2(\delta_1 - a)}} + \frac{1}{p^{\delta_2 - b+1}} + \frac{1}{p^{2(\delta_2 - b)}}\right)\right), \tag{4.4}$$

since the terms $\pm\frac{1}{p^{z_j - a}}$ and $\pm\frac{1}{p^{z_j - b}}$ cancel out. But, the Euler's product in (4.4) converges absolutely for any $\delta_1 > a + \frac{1}{2}$ and $\delta_2 > b + \frac{1}{2}$. Therefore, the identity (4.2) holds in the wider region (4.1). □

Now we write the multiple Dirichlet series expansion of the function $H(z_1, z_2, z_3, z_4)$ from Lemma 2:

$$H(z_1, z_2, z_3, z_4) = \sum_{n_1, n_2, n_3, n_4 = 1}^{\infty} \frac{h(n_1, n_2, n_3, n_4)}{n_1^{z_1} n_2^{z_2} n_3^{z_3} n_4^{z_4}}.$$

The function $h(n_1, n_2, n_3, n_4)$ defined in this way is also a multiplicative function of 4 variables. From the identity (4.2) we infer the following convolution identity between the corresponding multivariate arithmetic functions:

$$\varphi\left(\left[\frac{[n_1, n_2]^a}{(n_1, n_2)^c}, \frac{[n_3, n_4]^b}{(n_3, n_4)^d}\right]\right) = \sum_{j_1 d_1 = n_1, \ldots, j_4 d_4 = n_4} j_1^a j_2^a j_3^b j_4^b \quad h(d_1, d_2, d_3, d_4), \tag{4.5}$$

where the sum runs over all 4-tuples (j_1, j_2, j_3, j_4) in which j_i is a positive divisor of n_i, for all $1 \leq i \leq 4$.

Proof. (*of Theorem 2*) We start by employing the identity (4.5) in our summation function:

$$\sum_{n_1, n_2, n_3, n_4 \leq x} \varphi\left(\left[\frac{[n_1, n_2]^a}{(n_1, n_2)^c}, \frac{[n_3, n_4]^b}{(n_3, n_4)^d}\right]\right)$$

$$= \sum_{j_1 d_1 \leq x, \ldots, j_4 d_4 \leq x} j_1^a j_2^a j_3^b j_4^b \quad h(d_1, d_2, d_3, d_4)$$

$$= \sum_{d_1, d_2, d_3, d_4 \leq x} h(d_1, d_2, d_3, d_4) \sum_{j_1 \leq \frac{x}{d_1}} j_1^a \sum_{j_2 \leq \frac{x}{d_2}} j_2^a \sum_{j_3 \leq \frac{x}{d_3}} j_3^b \sum_{j_4 \leq \frac{x}{d_4}} j_4^b$$

$$= \sum_{d_1, d_2, d_3, d_4 \leq x} h(d_1, d_2, d_3, d_4) \left(\frac{x^{a+1}}{(a+1) d_1^{a+1}} + O\left(\frac{x^a}{d_1^a}\right)\right)$$

$$\times \left(\frac{x^{a+1}}{(a+1) d_2^{a+1}} + O\left(\frac{x^a}{d_2^a}\right)\right) \left(\frac{x^{b+1}}{(b+1) d_3^{b+1}} + O\left(\frac{x^b}{d_3^b}\right)\right) \left(\frac{x^{b+1}}{(b+1) d_4^{b+1}} + \left(\frac{x^b}{d_4^b}\right)\right)$$

$$= \frac{x^{2a+2b+4}}{(a+1)^2(b+1)^2} \sum_{d_1,d_2,d_3,d_4 \leq x} \frac{h(d_1,d_2,d_3,d_4)}{d_1^{a+1} d_2^{a+1} d_3^{b+1} d_4^{b+1}} + R(x). \tag{4.6}$$

Here, $R(x)$ is the remainder term, which is bounded by

$$R(x) \ll \sum_{\substack{u_1,u_2 \in \{a,a+1\} \\ v_1,v_2 \in \{b,b+1\} \\ (u_1,u_2,v_1,v_2) \neq \\ (a+1,a+1,b+1,b+1)}} x^{u_1+u_2+v_1+v_2} \sum_{d_1,d_2,d_3,d_4 \leq x} \frac{|h(d_1,d_2,d_3,d_4)|}{d_1^{u_1} d_2^{u_2} d_3^{v_1} d_4^{v_2}}, \tag{4.7}$$

where in the first summation at least one $u_i = a$, $i \in \{1,2\}$, or at least one $v_j = b$, $j \in \{1,2\}$. For one such 4-tuple, for example for $(u_1,u_2,v_1,v_2) = (a, a+1, b+1, b+1)$, the corresponding contribution on the right hand side of (4.7) is bounded by

$$\ll x^{2a+2b+3} \sum_{d_1,d_2,d_3,d_4 \leq x} \frac{|h(d_1,d_2,d_3,d_4)|}{d_1^{a} d_2^{a+1} d_3^{b+1} d_4^{b+1}}$$

$$= x^{2a+2b+3} \sum_{d_1,d_2,d_3,d_4 \leq x} \frac{|h(d_1,d_2,d_3,d_4)| d_1^{\frac{1}{2}+\epsilon}}{d_1^{a+\frac{1}{2}+\epsilon} d_2^{a+1} d_3^{b+1} d_4^{b+1}}$$

$$\leq x^{2a+2b+\frac{7}{2}+\epsilon} \sum_{d_1,d_2,d_3,d_4 \leq x} \frac{|h(d_1,d_2,d_3,d_4)|}{d_1^{a+\frac{1}{2}+\epsilon} d_2^{a+1} d_3^{b+1} d_4^{b+1}}, \tag{4.8}$$

for any $\epsilon > 0$. Here the 4-tuple of exponents $(a+\frac{1}{2}+\epsilon, a+1, b+1, b+1)$ belongs to the region of absolute convergence (4.1). Therefore, by Lemma 2 the multiple Dirichlet series (4.8) converges to a constant and hence we obtain the bound $O(x^{2a+2b+\frac{7}{2}+\epsilon})$. We can bound all the other terms in (4.7) similarly and we get

$$R(x) \ll x^{2a+2b+\frac{7}{2}+\epsilon}. \tag{4.9}$$

Finally, we return to the main term in (4.6). We have:

$$\sum_{d_1,d_2,d_3,d_4 \leq x} \frac{h(d_1,d_2,d_3,d_4)}{d_1^{a+1} d_2^{a+1} d_3^{b+1} d_4^{b+1}} = \sum_{d_1,d_2,d_3,d_4=1}^{\infty} \frac{h(d_1,d_2,d_3,d_4)}{d_1^{a+1} d_2^{a+1} d_3^{b+1} d_4^{b+1}}$$

$$- \sum_{\substack{I\subseteq\{1,2,3,4\}\\ I\neq\emptyset}} \sum_{\substack{d_i>x, i\in I\\ d_j\leq x, j\notin I}} \frac{h(d_1,d_2,d_3,d_4)}{d_1^{a+1}d_2^{a+1}d_3^{b+1}d_4^{b+1}}. \tag{4.10}$$

The complete multiple Dirichlet series in (4.10) converges by Lemma 2 and its sum is equal $H(a+1,a+1,b+1,b+1)$. All 15 terms for subsets $I\neq\emptyset$ can be bounded similarly. For illustration, we bound the contribution in (4.10) corresponding to $I=\{1,3\}$:

$$\sum_{\substack{d_1,d_3>x\\ d_2,d_4\leq x}} \frac{|h(d_1,d_2,d_3,d_4)|}{d_1^{a+1}d_2^{a+1}d_3^{b+1}d_4^{b+1}} = \sum_{\substack{d_1,d_3>x\\ d_2,d_4\leq x}} \frac{|h(d_1,d_2,d_3,d_4)|d_1^{-\frac{1}{2}+\epsilon}d_3^{-\frac{1}{2}+\epsilon}}{d_1^{a+\frac{1}{2}+\epsilon}d_2^{a+1}d_3^{b+\frac{1}{2}+\epsilon}d_4^{b+1}}$$

$$\leq x^{-1+2\epsilon} \sum_{d_1,d_2,d_3,d_4=1}^{\infty} \frac{|h(d_1,d_2,d_3,d_4)|}{d_1^{a+\frac{1}{2}+\epsilon}d_2^{a+1}d_3^{b+\frac{1}{2}+\epsilon}d_4^{b+1}}.$$

Here again the multiple Dirichlet series converges to a constant by Lemma 2, and we get the bound $O(x^{-1+2\epsilon})$. In general we get that the contribution of the terms corresponding to a subset $I\subseteq\{1,2,3,4\}$, $I\neq\emptyset$ is bounded by $O\left(x^{(-\frac{1}{2}+\epsilon)|I|}\right)$, where $|I|$ denotes the cardinality of the subset I. Therefore the total error obtained by completing the main term in (4.6) is $O(x^{2a+2b+\frac{7}{2}+\epsilon})$, i.e. it is the same as in (4.9). This finishes the proof of the required asymptotic formula with the constant $C_{a,c;b,d} = H(a+1,a+1,b+1,b+1)$. □

Remark 5. *Theorem 2 can be generalized by similar methods to other situations, for example for summation functions of arithmetic functions of the form*

$$(n_1,n_2,\ldots,n_{k+\ell+m}) \mapsto f\left(\left[\frac{[n_1,\ldots,n_k]^A}{(n_1,\ldots,n_k)^a}, \frac{[n_{k+1},\ldots,n_{k+\ell}]^B}{(n_{k+1},\ldots,n_{k+\ell})^b}, \frac{[n_{k+\ell+1},\ldots,n_{k+\ell+m}]^C}{(n_{k+\ell+1},\ldots,n_{k+\ell+m})^c}\right]\right)$$

for non-negative integers $A\geq a, B\geq b, C\geq c$ and for any complex valued multiplicative arithmetic functions f which for some real $r>0$ satisfy $|f(p)-p^r| = O(p^{r-\frac{1}{2}})$ for all primes p and $|f(p^\nu)| = O(p^{\nu r})$ for all p and all $\nu\geq 2$.

Examples of such functions are $n \mapsto n^r$, *the sum-of-divisors function* $\sigma_r(n) = \sum_{d|n} d^r$ *or the generalized Euler function* $\varphi_r(n) = \sum_{d|n} \mu(\frac{n}{d}) d^r$.

Table of Notation

The standard notations which are used throughout the text:

- $\mathbb{C}$: The complex plane.
- $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.
- $f * g$: the convolution product.
- $n = \prod_p p^{v_p(n)}$ is the prime power factorization of $n \in \mathbb{N}$, where $v_p(n)$ is the highest power of prime p dividing n, the product being over the primes p.
- $d|n$ means that d is a divisor of n.
- $d||n$ means that d is a unitary divisor of n, i.e., $d|n$ and $gcd(d, n/d) = 1$.
- $\sum_p f(p)$ and $\prod_p f(p)$ mean that the sum or the product is over all prime numbers.
- $\sum_{d|n} f(d)$ and $\prod_{d|n} f(d)$ mean that the sum or the product is over all positive divisors of n, including 1 and n.
- $f : \mathbb{N} \to \mathbb{C}$ is the arithmetic function.
- ζ is the Riemann zeta function.
- ψ is the chebyshev function.
- δ is the arithmetic function given by $\delta(1) = 1, \delta(n) = 0$ for $n > 1$.
- μ is the Möbius function.
- φ is the Euler's totient function.
- Λ is the von Mangoldt function.
- Ω is the total number of prime divisors of n.
- ω is the number of distinct prime divisors of n.
- $\sigma_k(n) = \sum_{d|n} d^k \quad (k \in \mathbb{C})$, is the sum of the k-th powers of the positive divisors of n including 1 and n.

- $\tau(n)$ is the number of divisors of n.
- $\sigma(n)$ is the sum of divisors of n.
- $\lambda(n) = (-1)^{\Omega(n)}$.
- $\mathcal{A}_k$ is the set of arithmetic functions of k variables ($k \in \mathbb{N}$), i.e., of functions $f : \mathbb{N} \to \mathbb{C}$.
- 1_k is the constant 1 function in $\mathcal{A}_k$, i.e., $1_k(n_1, \ldots, n_k) = 1$ for every $n_1, \ldots, n_k \in \mathbb{N}$.
- $\delta_k(n_1, \ldots, n_k) = \delta(n_1) \ldots \delta(n_k)$, that is $\delta_k(1, \ldots, 1) = 1$ and $\delta_k(n_1, \ldots, n_k) = 0$ for $n_1, \ldots, n_k > 1$.

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